Some Analytical Results for Models of the Bid-Ask Spread

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Abstract

The focus of this paper is on order processing models of the bid-ask spread, also termed as fixed-cost models. While other theories have been advanced to explain spreads, such as inventory holding costs and adverse selection, research indicates that the fixed cost component constitutes the bulk of the observed spread. This paper starts with the Roll model and the subsequent extension of Choi, Salandro and Shastri. It takes cognizance of the implications of such models for the observed stock prices and the mid-points of bid-ask quotes, to set up tests using the Generalized Method of Moments (GMM) technique. The paper develops an analytical variance-covariance matrix for the fixed cost model with instantaneous adjustment of prices to new information.

Keywords: Fixed Cost Models of the Bid-Ask spread, Quoted Spreads and Realized Spreads, Partial Adjustment and Overreaction of Prices, Generalized Method of Moments, Analytical Variance-Covariance Matrix, The Roll Model, The Choi Salandro and Shastri Extension

1. Introduction

The quoted bid-ask spread is defined as the difference between the price at which a dealer is willing to sell the security, the 'ask', and the price at which he is willing to buy it, the 'bid'. There are three major theories which purport to explain the existence of the spread. These are, order processing cost models, inventory cost models, and adverse selection models. Stoll (1989) develops a unified framework for studying the relative importance of these, and shows that the differences between the models lie in their predictions regarding the probability of a price reversal, and the magnitude of the price reversal.(Note 1) Stoll also makes a distinction between the quoted spread, and the effective or realized spread. The realized spread is defined as the difference between the price at which the dealer sells a security, and the price at which he buys at an earlier instant. The order processing models, which are the focus of this paper, predict that the two spreads are equal.

The simplest of the three models are the order processing, or fixed cost models. One of the most popular models in this class was developed by Roll (1984), and later extended by Choi, Salandro and Shastri (1988), hereafter referred to as CSS. Its premise is that the spread compensates the dealer for the costs incurred in processing buy and sell transactions. In this framework, transactions at the bid are offset by transactions at the ask, and the difference between the two is used by the dealer to defray his expenses.

The literature on the estimation of the realized spread is fairly diverse. Roll (1984) estimates realized spreads in an order processing framework using daily and weekly data and attempts to relate them to firm size. CSS (1988) use transaction data for stock options to estimate the spread, and the conditional probability of a price continuation. Stoll (1989) estimates the relative components of the spread using transaction prices and bid-ask quotes. Hasbrouck and Ho (1987) do not estimate spreads, but use transaction data and intraday quotes to show that observed stock returns may best be described by an ARMA(2,2) process. In addition to the spread, their model incorporates a delayed adjustment mechanism for observed prices. The partial adjustment model for the evolution of asset prices is also developed by Amihud and Mendelson (1987), who study the impact of the trading mechanism on the return generating process.

The objective of this paper is to study the order processing model. By focusing solely on the fixed cost model, we are ignoring the other two components of the spread, namely inventory and adverse selection costs. However, this does not imply that these costs are insignificant, although empirical research has shown that the fixed cost component constitutes the bulk of the spread (Note 2). The order processing model is often used by researchers who are forced to take the bid-ask spread into account as a 'nuisance' parameter, although they are not interested in the spread per se.

This paper takes advantage of the implications of the theoretical models for two series of observations, observed prices and the midpoints of bid-ask quotes, to set up tests with Hansen's (1982) Generalized Method of Moments (GMM) technique. Besides yielding a relation between the covariance of observed price changes and the spread, the fixed cost model, has testable implications for the midpoints of bid-ask quotes. The model also predicts that the variance of the observed price changes varies directly with the size of the spread. In addition to giving estimates of the realized spread, GMM facilitates the computation of a chi-squared statistic which can be used to judge the validity of the models. For the fixed cost model with instantaneous adjustment of prices to new information, we derive the analytical covariance matrix suggested by Richardson and Smith (1988).

In section two, we discuss the order processing cost model. In section three we develop the GMM test procedure for the models under consideration and describe the analytical covariance matrix for the estimates of the parameters of the Roll model. Finally, in section four we analyze the results and conclude the paper.

2. The Models

In this section we discuss the framework of the order processing model, under two different sets of assumptions regarding the price adjustment process. We first study the Roll model using notation that is identical to that used by Harris (1989b). In this context, we also briefly examine an extension of the original model proposed by CSS. Then, we consider a fixed cost model under the assumption that prices need not always instantaneously adjust to new information. This added flexibility allows prices to either partially adjust to or over-react to information.

2.1 The Roll Model

The order processing model was developed under the following set of assumptions.

1. Security prices at any given time, fully incorporate all relevant available information. What this means is that the price that would be observed in the absence of market imperfections accurately reflects the intrinsic value of the asset.

2. Buy or sell orders are equally probable at any instant in time, on an unconditional basis, and order flows are serially independent.

3. The underlying price of the asset is independent of the order flow in the market. This implicitly rules out any adverse information effects.

4. The true value of the security is bracketed by the bid-ask spread and the spread is assumed to be constant and symmetric (Note 3), at least for the time period for which the analysis is conducted.

Mathematically, the model may be stated as follows.

$$P_t = P_t^* + \frac{s}{2}Q_t \tag{1}$$

and,

$$P_t^* = P_{t-1}^* + \mu + \epsilon_t, \tag{2}$$

where P_t is the observed price at time t, P_t^* is the intrinsic value of the asset at time t, which would have been observed in the absence of the spread, and Q_t is the indicator of the transaction type. $Q_t = 1$ if the observed price is at the ask, and -1 if it is at the bid. The value innovations, $\{\epsilon\}$ are assumed to be serially uncorrelated, and have a zero mean and

constant variance σ_ϵ^2. (Note 4) The drift in the true prices μ is assumed to be constant over time, as is the spread s.

With these assumptions, the covariance of the observed price changes equals minus the square of half the bid-ask spread, i.e., $Cov(\triangle P_t, \triangle P_{t-1}) = -s^2/4$, and the estimates of the covariance can be transformed to get the spreads. (Note 5)

2.1.1 The CSS Extension

CSS generalize the model by partially relaxing the second assumption made above. In their model, although the unconditional probability of a bid or an ask transaction is the same at a particular point in time, conditional on a bid, the probability of a consecutive bid is greater than a half, and likewise for the ask. In other words, transaction types are positively serially correlated. A major difference between the two models, is that in the CSS framework, the interval at which successive transactions are measured is important, while it is irrelevant for the Roll model.

There are two reasons as to why transaction types may be serially correlated in practice. First, large market orders may be split up by floor brokers, thereby ensuring that successive transactions take place on the same side of the market. Similarly, a movement in the price will cause limit orders on the same side of the order book to be executed in succession.

In Appendix II we show that even if the conditional probability of a bid following a bid or that of an ask following an ask, is greater than half, the formula for the spread converges to that predicted by the Roll model, if we consider time intervals with an adequate number of transactions. Assuming that transactions occur at fixed intervals of time, we show that if there are five transactions in the time interval, then the formula converges even if the probability is as high as 0.75. Thus, if we assume that the stock trades at one minute intervals, we can use five minute intervals to estimate the spread.

2.2 A Fixed Cost Model With Partial Adjustment/Over-reaction of Prices to Information

In this subsection, we develop a model for observed prices by combining the features of the fixed cost models proposed by Roll and Harris with the partial adjustment mechanism for prices developed by Amihud and Mendelson and Hasbrouck and Ho. The term 'partial adjustment' is a misnomer, for, the model also allows the prices to overreact to new information.

The assumptions under which this model is developed may be stated as follows.

1. The intrinsic or true value of an asset follows a random walk. However, even in the absence of market frictions like bid-ask spreads, the intrinsic values need not be observable in the market. Under such conditions, the market will observe an adjusted price series, where the adjusted price at a point in time either partially incorporates the innovation in the intrinsic value or else represents an overreaction to the new information. The full adjustment model is a special case, when adjusted prices reflect all available information and hence are equivalent to the intrinsic values.

2. The probability of observing a buy or a sell order is unconditionally the same at a point in time, and order flows are serially independent.

3. Both the intrinsic value as well as the adjusted price of an asset are independent of the order flow in the market.

4. The adjusted security price is bracketed by the bid-ask spread and the spread is constant and symmetric.

The model may be mathematically expressed as follows. Let the true price series $\{V\}$ be given by,

$$V_t = V_{t-1} + \mu + \epsilon_t. \tag{3}$$

The drift μ is assumed to be a constant and the innovations $\{\epsilon\}$ follow a stationary distribution with mean zero and variance $= \sigma_\epsilon^2$. The adjusted price series $\{P^*\}$ may be expressed as,

$$P_t^* = P_{t-1}^* + g(V_t - P_{t-1}^*). \tag{4}$$

'g' is the adjustment coefficient. If g=1, $P_t^* = V_t$, and prices incorporate all available information. A value of g such that $0 < g < 1$ represents partial adjustment of prices, while if $g > 1$, we have over-reaction of prices to information. In order to ensure that the variances of the changes in the adjusted and observed prices are positive, we are forced to restrict g to have a value less than two. The observed price series $\{P\}$ is as before given by,

$$P_t = P_t^* + \frac{s}{2}Q_t, \tag{5}$$

where Q_t is the indicator of the transaction type. The only difference from the model studied earlier is that P^*, or the quote midpoint, need no longer be the true price and need not follow a random walk.

Our formulation of the partial adjustment feature is identical to the model studied by Hasbrouck and Ho, but differs slightly from the model proposed by Amihud and Mendelson. The latter specify a random walk for the intrinsic values as shown in equation 3.

However, their adjusted and observed prices are identical, and are given by,

$$P_t = P_{t-1} + g(V_t - P_{t-1}) + u_t, \tag{6}$$

where $\{u\}$ is white noise with zero mean and a finite variance. The noise is a result of noise trading as well as the impact of market microstructure effects. In this set up, the observed price is affected by innovations in the intrinsic value and by current and past noise trading and market microstructure effects.

In our model, since the only source of friction is the spread which is assumed to be a constant, we do not want the bid and ask quotes to change except due to the adjustments caused by the innovations in the intrinsic value. Hence, we make a distinction between adjusted prices and observed prices. Adjusted prices reflect the impact of the value innovations, while the observed prices differ from the adjusted prices solely because of a constant bid-ask spread.

3. The Estimation Procedure

The models that have been considered thus far lend themselves to estimation and testing using Hansen's GMM procedure. In this section we derive the population moment conditions for the Roll model. In addition, for the Roll model, we test the hypothesis that the value innovations, $\{\epsilon\}$, are drawn from a normal distribution, and also derive the analytical variance-covariance matrix for the unknown parameters.

3.1 The Roll Model

Given the assumptions of the model and equations (1) and (2), a number of results may be derived. Among them are the following.

$$E(\triangle P_t^*) = \mu = E(\triangle P_t) \tag{7}$$

The expected change in the true price of the security is a constant, and is equal to the expected change in the observed price of the security.

$$Var(\triangle P_t^*) = \sigma_\epsilon^2. \tag{8}$$

The variance of changes in the true price of the security is a constant.

$$Var(\triangle P_t) = \sigma_\epsilon^2 + s^2/2. \tag{9}$$

The variance of changes in the observed price of the asset is equal to the variance of true price changes plus one half the square of the bid-ask spread.

$$Cov(\triangle P_t^*, \triangle P_{t-1}^*) = 0. \tag{10}$$

The first order serial covariance of true price changes is zero.

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2/4. \tag{11}$$

The first order serial covariance of observed price changes is minus one fourth the square of the bid-ask spread.

The true price, P^*, is not directly observable. However, the ask at time 't' is given by,

$$P_{at} = P_t^* + s/2, \tag{12}$$

while the bid is given by,

$$P_{bt} = P_t^* - s/2. \tag{13}$$

Therefore, $(P_{at} + P_{bt})/2 = P_t^*$, and hence, by using the midpoints of the bid and the ask quotes, we can estimate the mean and the variance of the 'true' price series, assuming that the spread is symmetric.

We have six moment conditions, and only three unknowns, μ, σ_ϵ^2 and s^2. Additional restrictions may be generated by using a longer measurement interval. For example,

$$Var(P_t^* - P_{t-k}^*) = k\sigma_\epsilon^2, \tag{14}$$

and we can get one such expression for every value of k we choose. Typically, these additional restrictions entail the use of overlapping observations when the model is tested. (Note 6) However, to keep the analysis simple, we focus only on the six equations discussed above.

The population moment conditions may be jointly expressed as,

$$E\begin{pmatrix} \bar{P}_t - \bar{P}_{t-1} - \mu \\ P_t - P_{t-1} - \mu \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)^2 - \sigma_\epsilon^2 \\ (P_t - P_{t-1} - \mu)^2 - \sigma_\epsilon^2 - s^2/2 \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)(\bar{P}_{t-1} - \bar{P}_{t-2} - \mu) \\ (P_t - P_{t-1} - \mu)(P_{t-1} - P_{t-2} - \mu) + s^2/4 \end{pmatrix} = 0, \tag{15}$$

where the P's are the observed transaction prices, and the $\bar{P}$'s the average of the observed bid and ask quotes.

We derive the analytical variance-covariance matrix for the above moment conditions in Appendix I. It can be seen that the first two moment conditions have identical variances, and the same covariances with all the other restrictions, and the matrix is therefore singular. In order to overcome this problem, we drop the first moment condition. In a finite sample, we apply the test to,

$$g_T(\mu, \sigma_\epsilon^2, s^2) = \frac{1}{T} \sum_{t=1}^{T} \begin{pmatrix} P_t - P_{t-1} - \mu \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)^2 - \sigma_\epsilon^2 \\ (P_t - P_{t-1} - \mu)^2 - \sigma_\epsilon^2 - s^2/2 \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)(\bar{P}_{t-1} - \bar{P}_{t-2} - \mu) \\ (P_t - P_{t-1} - \mu)(P_{t-1} - P_{t-2} - \mu) + s^2/4 \end{pmatrix} \tag{16}$$

The overidentifying statistic associated with this equation has an asymptotic chi-squared distribution, with two degrees of freedom. That is,

$$T g_T' W_0 g_T \sim \chi^2(2), \tag{17}$$

where W_0 is the inverse of the variance-covariance matrix of restrictions, and

$$\sqrt{T}(\hat{\theta} - \theta) \sim N[0, (D'WD)^{-1}] \equiv N[0, \begin{pmatrix} \sigma_\epsilon^2 & m_3 & 0 \\ m_3 & m_4 - \sigma_\epsilon^4 & 0 \\ 0 & 0 & \frac{s^6 + 40s^4\sigma_\epsilon^2 + 112s^2\sigma_\epsilon^4}{2s^2 + 16\sigma_\epsilon^2} \end{pmatrix}], \tag{18}$$

where m_i is the 'i'th central moment of $\{\epsilon\}$.

3.1.1 Testing for Normality

In equation (2), if we make the assumption that the innovations in the true price series $\{\epsilon\}$ are normally distributed with mean zero and variance σ_ϵ^2 then,

$$\triangle P^* \sim N(\mu, \sigma_\epsilon^2). \tag{19}$$

If $\triangle P^*$ is normally distributed it can be shown that $| \triangle P^* |$ follows a chi distribution with one degree of freedom, and noncentrality parameter $= (\mu/\sigma)^2$. It can also be shown that, (Note 7)

$$E \mid \triangle P_t^* \mid = (\frac{2}{\pi})^{1/2}\sigma exp(\frac{-1}{2}\frac{\mu^2}{\sigma^2}) + \mu[1 - 2N(\frac{-\mu}{\sigma})], \tag{20}$$

where N is the standard normal distribution function.

From equation (1) we see that $\triangle P_t = \triangle P_t^* + \frac{s}{2}\triangle Q_t$. Thus, if $\triangle P^*$ follows a normal distribution, then conditional on $\triangle Q$, $\triangle P$ will also follow a normal distribution. Hence, $| \triangle P | \triangle Q |$ will follow a chi distribution. Since $\triangle Q$ can take on only three possible values, we can easily compute the conditional expectations of $| \triangle P |$ and weight them in order to get the unconditional expectation of $| \triangle P |$. In this case it can be shown that,

$$E \mid \triangle P_t \mid = (\frac{2}{\pi})^{1/2}\sigma\{\frac{1}{4}exp(\frac{-1}{2}\frac{(\mu - s)^2}{\sigma^2}) + \frac{1}{4}exp(\frac{-1}{2}\frac{(\mu + s)^2}{\sigma^2}) + \frac{1}{2}exp(\frac{-1}{2}\frac{\mu^2}{\sigma^2})\}$$

$$+ \frac{1}{4}(\mu - s)[1 - 2N(\frac{-(\mu - s)}{\sigma})] + \frac{1}{4}(\mu + s)[1 - 2N(\frac{-(\mu + s)}{\sigma})]$$

$$+ \frac{1}{2}(\mu)[1 - 2N(\frac{-(\mu)}{\sigma})]. \tag{21}$$

Equations (20) and (21) can be combined with the five moment conditions obtained earlier in order to test the hypothesis that the price innovations are normally distributed (Note 8). We will then have seven equations with three unknowns, and the overidentifying chisquared statistic will have four degrees of freedom.

3.2 A Fixed Cost Model With Partial Adjustment/Overreaction

The population moment conditions for this model can be derived using equations (3), (4) and (5). These conditions may be stated as follows.

$$E(\triangle P_t^*) = \mu = E(\triangle P_t) \tag{22}$$

This result is similar to that obtained for the Roll model, namely, the expected change in the adjusted price is a constant and is equal to the expected change in the observed price.

$$Var(\triangle P_t^*) = \frac{g}{(2 - g)}\sigma_\epsilon^2. \tag{23}$$

The variance of changes in the adjusted price is a constant, and is a function of the variance of the innovations in the intrinsic value as well as the adjustment coefficient. If g = 1, prices adjust fully and the variance is the same as that obtained for the Roll model. In order for the variance to be positive we require that g be less than two.

$$Var(\triangle P_t) = \frac{g}{(2 - g)}\sigma_\epsilon^2 + s^2/2. \tag{24}$$

The variance of changes in the observed price of the asset is equal to the variance of adjusted price changes plus one half the square of the bid-ask spread.

$$Cov(\triangle P_t^*, \triangle P_{t-1}^*) = \frac{g(1-g)}{(2-g)}\sigma_\epsilon^2 \tag{25}$$

The first order serial covariance of the changes in the adjusted price series is in general not equal to zero, and depends on g and σ_ϵ^2.

$$Cov(\triangle P_t, \triangle P_{t-1}) = \frac{g(1-g)}{(2-g)}\sigma_\epsilon^2 - s^2/4. \tag{26}$$

The first order serial covariance of observed price changes is equal to the first order serial covariance of changes in the quote midpoints minus one fourth the square of the bid-ask spread. The serial covariance need not always be negative. If prices overreact or adjust fully, then the covariance will be negative. With partial adjustment, however, the covariance may be either positive or negative.

The population moment conditions may be jointly expressed as,

$$E\begin{pmatrix} \bar{P}_t - \bar{P}_{t-1} - \mu \\ P_t - P_{t-1} - \mu \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)^2 - \sigma_\epsilon^2 \frac{g}{(2-g)} \\ (P_t - P_{t-1} - \mu)^2 - \sigma_\epsilon^2 \frac{g}{(2-g)} - s^2/2 \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)(\bar{P}_{t-1} - \bar{P}_{t-2} - \mu) - \sigma_\epsilon^2 \frac{g(1-g)}{(2-g)} \\ (P_t - P_{t-1} - \mu)(P_{t-1} - P_{t-2} - \mu) - \sigma_\epsilon^2 \frac{g(1-g)}{(2-g)} + s^2/4 \end{pmatrix} = 0, \tag{27}$$

where the P's are the transaction prices, and the $\bar{P}$'s the averages of the bid and ask quotes.

In order to be consistent with the approach for the Roll model, we will drop the first moment condition and, in a finite sample, apply the test to,

$$g_T(\mu, \sigma_\epsilon^2, s^2, g) =$$

$$\frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} P_t - P_{t-1} - \mu \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)^2 - \sigma_\epsilon^2 \frac{g}{(2-g)} \\ (P_t - P_{t-1} - \mu)^2 - \sigma_\epsilon^2 \frac{g}{(2-g)} - s^2/2 \\ (\bar{P}_t - \bar{P}_{t-1} - \mu)(\bar{P}_{t-1} - \bar{P}_{t-2} - \mu) - \sigma_\epsilon^2 \frac{g(1-g)}{(2-g)} \\ (P_t - P_{t-1} - \mu)(P_{t-1} - P_{t-2} - \mu) - \sigma_\epsilon^2 \frac{g(1-g)}{(2-g)} + s^2/4 \end{pmatrix} = 0. \tag{28}$$

We have five equations and four unknowns, and the overidentifying chi-squared statistic has one degree of freedom.

4. Conclusions and Testable Implications

In this paper, we derive testable implications of the theoretical models for two series of observations, observed prices and the midpoints of bid-ask quotes, using a Generalized Method of Moments (GMM) technique. The fixed cost model, has testable implications for testing the relationship between the covariance of observed price changes and the spread. In addition, it also has implications for the mid-points of bid-ask quotes. In addition to giving estimates of the realized spread, GMM facilitates the computation of a chi-squared statistic which can be used to judge the validity of the models.

The Roll model predicts that the first order serial covariance of observed price changes will always be negative. Hence an estimation of consistently positive values, would serve to reject this model. It also predicts that the estimates of the spread should not change, when we change the intervals at which trade prices and bid-ask quotes are observed. The partial adjustment model, however allows the first order serial covariances of price changes to be negative as well as positive. It postulates that the first order serial covariance is a function of the spread as well as the adjustment factor. If the factor is greater than or equal to one, the covariance will be negative. However, if the adjustment factor is less than one, the covariance may be positive or negative. Just as in the case of the full adjustment model, the partial adjustment

factor also requires the estimate of the spread to be independent of the time interval in which prices and quotes are observed.

References

Amihud Y., & Mendelson H. (1987). Trading mechanisms and stock returns: An empirical investigation. *Journal of Finance,* 533-553.

Choi, J. Y., Salandro, D., & Shastri, K. (1988). On the estimation of bid-ask spreads: Theory and evidence. *Journal of Financial and Quantitative Analysis*, 219-230.

Foster, D. F., & Viswanathan, S. (1990). *Variations in volumes, variances and trading costs* (Unpublished Manuscript, Duke University).

Hansen L.P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica*, 1029-1054. https://doi.org/10.2307/1912775

Harris L. (1989). *The power of the Roll serial covariance bid/ask spread estimator* (Unpublished Manuscript).

Hasbrouck, J., & Ho, T. S. Y. (1987) Order arrival, quote behaviour, and the return generating process. *Journal of Finance*, 1035-1048.

Leone, F. C., Nelson, L. S., & Nottingham, R. B. (1961). The folded normal distribution. *Technometrics*, 543-550.

Lo, A. W., & MacKinlay, A. C. (1988). Stock market prices do not follow random walks: Evidence from a simple specification test. *The Review of Financial Studies*, 41-66.

Richardson, M., & Smith, T. (1988). *Tests of financial models in the presence of overlapping observations* (Unpublished Manuscript, Stanford University).

Roll, R. (1984). A simple implicit measure of the effective bid-ask spread in an efficient market. *Journal of Finance*, 1127-1139.

Stoll, H. (1989). Inferring the components of the bid-ask spread: Theory and empirical tests. *Journal of Finance*.

Notes

Note 1. Foster and Viswanathan (1990) also study the three components of the spread simultaneously. The major focus of their paper is the adverse selection costs.

Note 2. See Foster and Viswanathan (1990).

Note 3. Roll (1984) has shown that the size of the spread can be allowed to depend on the magnitude of price changes, without changing the structure of the model. If so, the estimated realized spread is the average spread over the sample period.

Note 4. Researchers often assume that the $\{\epsilon\}$'s are i.i.d. This, however, is not necessary for our study.

Note 5. The resulting estimate is subject to underestimation due to Jensen's inequality, because the transformation uses a concave function.

Note 6. See Lo and MacKinlay (1988), and Richardson and Smith (1988).

Note 7. See Leon, Nelson, and Nottingham (1961).

Note 8. This is a convenient test for normality, that can be easily carried out in a method of moments framework. It is possible that there are other, more powerful, tests which can be used as well.

Appendix I

The Analytical Variance Covariance Matrix of the Moment Conditions.

From equation (4.8), we have six moment conditions involving P_t and $\bar{P}_t$. Under the null hypothesis, $\bar{P}_t \equiv P_t^*$, the true price. Therefore (4.8) may be expressed as,

$$w_0 = \begin{pmatrix} \epsilon_t \\ \epsilon_t + \frac{s}{2}\triangle Q_t \\ \epsilon_t^2 - \sigma_\epsilon^2 \\ (\epsilon_t + \frac{s}{2}\triangle Q_t)^2 - \sigma_\epsilon^2 - s^2/2 \\ \epsilon_t \epsilon_{t-1} \\ (\epsilon_t + \frac{s}{2}\triangle Q_t)(\epsilon_{t-1} + \frac{s}{2}\triangle Q_{t-1}) + s^2/4 \end{pmatrix}.$$

The model assumes that (ϵ) and (Q) are independent. Therefore, $Ef(\epsilon)g(Q) = Ef(\epsilon)Eg(Q)$, where f and g are functions of ϵ and Q respectively. We have also assumed that,

$$E(\epsilon_t) = 0,$$

$$E(\epsilon_t^2) = \sigma_\epsilon^2,$$

and

$$E(\epsilon_t \epsilon_{t-j}) = 0, j \neq 0.$$

In order to derive the analytical variance covariance matrix, it is sufficient to assume that the (ϵ)'s are independently and identically distributed, but, this is not necessary. For our application, we need to only impose the following restrictions on the (ϵ)'s:

$$E(\epsilon_t^2 \epsilon_{t-j}) = 0, j \neq 0.$$

$$E(\epsilon_t^3) = m_3,$$

and m_3 is finite.

$$E(\epsilon_t^3 \epsilon_{t-1}) = 0.)$$

$$E(\epsilon_t^4) = m_4,$$

and m_4 is finite.

$$E(\epsilon_t^2 \epsilon_{t-j}^2) = \sigma_\epsilon^4, j \neq 0.$$

$$E(\epsilon_t \epsilon_{t-j} \epsilon_{t-j-1}) = 0.$$

$$E(\epsilon_t^2 \epsilon_{t-j} \epsilon_{t-j-1}) = 0.$$

$$E(\epsilon_t \epsilon_{t-1} \epsilon_{t-j} \epsilon_{t-j-1}) = 0.$$

We can show that,

$$E(\triangle Q_t) = 0, \qquad\qquad\qquad (13)$$

$$E(\triangle Q_t^2) = 2,$$

and

$$E(\triangle Q_t \triangle Q_{t-1}) = -1.$$

We can also demonstrate the following, for $\triangle Q$.

$$E(\triangle Q_t \triangle Q_{t-j}) = 0, j \neq 0, \pm 1.$$

$$E(\triangle Q_t^2 \triangle Q_{t-j}) = 0, \forall j.$$

$$E(\triangle Q_t^2 \triangle Q_{t-j}^2) = 4, j \neq 0.$$

$$E(\triangle Q_t^4) = 8.$$

$$E(\triangle Q^3_{t-j}\triangle Q_t) = -4, j = \pm 1.$$

$$E(\triangle Q^3_{t-j}\triangle Q_t) = 0, j \neq 0, \pm 1.$$

$$E(\triangle Q_{t-j-1}\triangle Q_{t-j}\triangle Q_t) = 0, \forall j.$$

$$E(\triangle Q^2_{t-j-1}\triangle Q_{t-j}\triangle Q_t) = -2, j = +1.$$

$$E(\triangle Q^2_{t-j-1}\triangle Q_{t-j}\triangle Q_t) = 0, j \neq 0, +1.$$

$$E(\triangle Q_{t-j-1}\triangle Q_{t-j}\triangle Q^2_t) = -2, j \neq 0, -1.$$

$$E(\triangle Q_{t-j-1}\triangle Q^2_{t-j}\triangle Q_t) = 2, j = 1.$$

$$E(\triangle Q_{t-j-1}\triangle Q^2_{t-j}\triangle Q_t) = 0, j \neq 0, \pm 1.$$

$$E(\triangle Q_{t-j-1}\triangle Q_{t-j}\triangle Q_{t-1}\triangle Q_t) = 1, j \neq 0, \pm 1.$$

The analytical variance-covariance matrix $S_0 = \sum_{l=-\infty}^{l=\infty} E(w_o w_l\prime)$. For the model that we are considering, it can be shown that the only lags which matter are $l = \pm 1$. Using the assumptions and the results that we have established, the analytical variance-covariance matrix for the six moment restrictions shown in equation (4.8) can be shown to be,

$$S_0 = \begin{pmatrix} \sigma_\epsilon^2 & \sigma_\epsilon^2 & m_3 & m_3 & 0 & 0 \\ \sigma_\epsilon^2 & \sigma_\epsilon^2 & m_3 & m_3 & 0 & 0 \\ m_3 & m_3 & m_4 - \sigma_\epsilon^4 & m_4 - \sigma_\epsilon^4 & 0 & 0 \\ m_3 & m_3 & m_4 - \sigma_\epsilon^4 & \frac{s^4}{4} + 2s^2\sigma_\epsilon^2 - \sigma_\epsilon^4 + m_4 & 0 & -\frac{s^4}{4} - \frac{s^2}{2}\sigma_\epsilon^2 \\ 0 & 0 & 0 & 0 & \sigma_\epsilon^4 & \sigma_\epsilon^4 \\ 0 & 0 & 0 & -\frac{s^4}{4} - \frac{s^2}{2}\sigma_\epsilon^2 & \sigma_\epsilon^4 & \frac{5s^4}{16} + \sigma_\epsilon^2 s^2 + \sigma_\epsilon^4 \end{pmatrix}.$$

Since this matrix is singular, we drop the first moment condition. The covariance matrix for the reduced set of conditions is,

$$S_0 = \begin{pmatrix} \sigma_\epsilon^2 & m_3 & m_3 & 0 & 0 \\ m_3 & m_4 - \sigma_\epsilon^4 & m_4 - \sigma_\epsilon^4 & 0 & 0 \\ m_3 & m_4 - \sigma_\epsilon^4 & \frac{s^4}{4} + 2s^2\sigma_\epsilon^2 - \sigma_\epsilon^4 + m_4 & 0 & -\frac{s^4}{4} - \frac{s^2}{2}\sigma_\epsilon^2 \\ 0 & 0 & 0 & \sigma_\epsilon^4 & \sigma_\epsilon^4 \\ 0 & 0 & -\frac{s^4}{4} - \frac{s^2}{2}\sigma_\epsilon^2 & \sigma_\epsilon^4 & \frac{5s^4}{16} + \sigma_\epsilon^2 s^2 + \sigma_\epsilon^4 \end{pmatrix}.$$

The derivative matrix is given by,

$$D_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

The variance-covariance matrix of the three unknowns, μ, σ^2, and s^2 is given by $(D_0\prime S_0^{-1} D_0)^{-1}$.

Appendix II

Serial Covariance of Price Changes with Serially Correlated Transaction Types.

We derive an expression for the spread, when 'δ', the conditional probability of a bid following a bid, or of an ask following an ask, is not equal to 1/2. The covariance, in this case, depends on the length of the measurement interval. More specifically, it depends on 'N', where 'N' is the number of transactions that have occurred between t-1 and t. For the purpose of this analysis, we will assume that transactions occur at fixed intervals in time,i.e., N is fixed for a given measurement interval.

a) N=1: In this case, the joint probability distribution can be shown to be,

		$\triangle P_t$		
		-s	0	s
	-s	0	$\delta(1-\delta)/2$	$(1-\delta)^2/2$
$\triangle P_{t-1}$	0	$\delta(1-\delta)/2$	δ^2	$\delta(1-\delta)/2$
	s	$(1-\delta)^2/2$	$\delta(1-\delta)/2$	0

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2(1-\delta)^2.$$

b) N=2: In this case, the joint probability distribution can be shown to be,

$\triangle P_{t-1}$	$\triangle P_t$		
	-s	0	s
-s	0	$\delta(1-\delta)[\delta^2+(1-\delta)^2]$	$2\delta^2(1-\delta)^2$
0	$[\delta^2+(1-\delta)^2]\times$		$[\delta^2+(1-\delta)^2]\times$
	$\delta(1-\delta)$	$[\delta^2+(1-\delta)^2]^2$	$\delta(1-\delta)$
s	$2\delta^2(1-\delta)^2$	$[\delta^2+(1-\delta)^2]\delta(1-\delta)$	0

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2[2(1-\delta)\delta]^2.$$

From the above distributions, two main features of the covariance are obvious.

i) The only two entries which affect the covariance are (s,-s) and (-s,s).
ii) Because of the symmetric nature of the model, the two probabilities are equal. Hence, we need only focus on the probability of the sequence bid-ask-bid, in order to derive the covariance.
c) N=3:

$$Pr(s,-s) = \frac{1}{2}[3\delta^2(1-\delta)+(1-\delta)^3]^2$$

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2[3\delta^2(1-\delta)+(1-\delta)^3]^2$$

d) N=4:

$$Pr(s,-s) = \frac{1}{2}[4\delta^3(1-\delta)+4\delta(1-\delta)^3]^2$$

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2[4\delta^3(1-\delta)+4\delta(1-\delta)^3]^2$$

e) N=5:

$$Pr(s,-s) = \frac{1}{2}[5\delta^4(1-\delta)+10\delta^2(1-\delta)^3+(1-\delta)^5]^2$$

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2[5\delta^4(1-\delta)+10\delta^2(1-\delta)^3+(1-\delta)^5]^2$$

In general,

$$Cov(\triangle P_t, \triangle P_{t-1}) = -s^2 \left[\sum_{k=1}^{N} C_{2k-1}^{N} \delta^{N-(2k-1)}(1-\delta)^{2k-1} \right]^2$$

where $C_b^a = 0$ if $a < b$, else $C_b^a = \frac{a!}{(a-b)!b!}$.

For a given value of $Cov(\triangle P_t, \triangle P_{t-1})$, we will now compare the values of the spread obtained using Roll's formula and the exact formula.

$\delta = .60$		
N	Roll	Exact
1	$2(-c)^{1/2}$	$2.5(-c)^{1/2}$
2	$2(-c)^{1/2}$	$2.083(-c)^{1/2}$
3	$2(-c)^{1/2}$	$2.016(-c)^{1/2}$
4	$2(-c)^{1/2}$	$2.003(-c)^{1/2}$
5	$2(-c)^{1/2}$	$2.001(-c)^{1/2}$
10	$2(-c)^{1/2}$	$2.000(-c)^{1/2}$

$\delta = .75$		
N	Roll	Exact
1	$2(-c)^{1/2}$	$4(-c)^{1/2}$
2	$2(-c)^{1/2}$	$2.667(-c)^{1/2}$
3	$2(-c)^{1/2}$	$2.286(-c)^{1/2}$
4	$2(-c)^{1/2}$	$2.133(-c)^{1/2}$
5	$2(-c)^{1/2}$	$2.065(-c)^{1/2}$
10	$2(-c)^{1/2}$	$2.002(-c)^{1/2}$

$\delta = .90$		
N	Roll	Exact
1	$2(-c)^{1/2}$	$10(-c)^{1/2}$
2	$2(-c)^{1/2}$	$5.556(-c)^{1/2}$
3	$2(-c)^{1/2}$	$4.098(-c)^{1/2}$
4	$2(-c)^{1/2}$	$3.388(-c)^{1/2}$
5	$2(-c)^{1/2}$	$2.445(-c)^{1/2}$
10	$2(-c)^{1/2}$	$2.241(-c)^{1/2}$

As we can see, the exact formula converges to Roll's formula as N increases. Thus, even for a high value of δ, like .75, if the stock trades every minute, Roll's formula will be fairly accurate if 5 minute price changes are used.