ORIGINAL RESEARCH

The nonmonotonic Gentzen deduction systems for the propositional logic

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ABSTRACT

The traditional propositional logic is monotonic. With the same logical language as, the same valuation as and the validity of a sequent different from the traditional propositional logic, a propositional logic could be nonmonotonic. In this paper, the four Gentzen deduction systems G^1, G^2, G^3, G^4 and their dualities G_1, G_2, G_3, G_4 will be given which are proved to be sound and complete with respect to the four definitions and their dualities of the validity of sequents, among which one is traditional and others are variations of the traditional one. Moreover, G^1, G^3 are monotonic in both Γ and Δ; and G^2, G^4 are monotonic in Γ and nonmonotonic in Δ. Dually, G_1, G_3 are nonmonotonic in both Γ and Δ; and G_2, G_4 are nonmonotonic in Γ and monotonic in Δ.

Key Words: Propositional logic, The Gentzen deduction system, Validity, Sequent, Soundness, Completeness

1. INTRODUCTION

The traditional Gentzen deduction system G^1 for the propositional logic is monotonic,[1,2] that is, given any theories $\Gamma, \Gamma', \Delta, \Delta'$, if sequent $\Gamma \Rightarrow \Delta$ is provable in G^1 and $\Gamma \subseteq \Gamma', \Delta \subseteq \Delta'$ then $\Gamma' \Rightarrow \Delta, \Gamma \Rightarrow \Delta', \Gamma' \Rightarrow \Delta'$ are provable in G^1. Correspondingly, the Gentzen deduction system G_1 for $\Gamma \not\Rightarrow \Delta$ is nonmonotonic,[3–5] that is, given any theories $\Gamma, \Gamma', \Delta, \Delta'$, if sequent $\Gamma \not\Rightarrow \Delta$ is provable in G_1 and $\Gamma \subseteq \Gamma', \Delta \subseteq \Delta'$ then it may be true that $\Gamma' \not\Rightarrow \Delta, \Gamma \not\Rightarrow \Delta', \Gamma' \not\Rightarrow \Delta'$ are provable in G_1.

The nonmonotonic logics are different from the monotonic logics in that the deduction is nonmonotonic. The traditional logics, such as the propositional logic, the first-order logic, modal logic, etc., are monotonic. The nontraditional logics, such as the default logic, the autoepistemic logic, circumscription, *etc.*, are nonmonotonic.[4–8]

The nonmonotonicity of a nonmonotonic logic follows from using a negation $\Delta \nvdash A$ of a monotonic deduction $\Delta \vdash A$. We found that each nonmonotonic logic has the occurrence of $\Delta \nvdash A$. For example, a formula B is deducible in the default logic (or in some extension of default theory $(\Delta, \{\frac{A:B}{B}\})$ from a default theory $(\Delta, \{\frac{A:B}{B}\})$ if A is deducible in propositional logic from Δ and $\neg B$ is not, that is,

$$\Delta \vdash A \& \Delta \nvdash \neg B.$$

It is obvious that the monotonicity of $\Delta \vdash A$ implies the nonmonotonicity of $\Delta \nvdash A$.

As a deduction relation, $\nvdash$ is contradictory to $\vdash$. Correspondingly, in the Gentzen deduction systems, the validity of $\Gamma \Rightarrow \Delta$ is contradictory to the invalidity of $\Gamma \Rightarrow \Delta$, *i.e.*, the validity of $\Gamma \mapsto \Delta$, where $\Gamma \mapsto \Delta$ is valid if there is an

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assignment v such that v satisfies Γ and does not satisfy Δ, where v satisfies Γ if v satisfies each formula in Γ; and v satisfies Δ if v satisfies some formula in Δ.

Therefore, as a contradictory relation $\vdash \Gamma \mapsto \Delta$ of $\vdash \Gamma \Rightarrow \Delta$, there is a Gentzen-typed deduction system G_1 such that G_1 is sound and complete, that is, for any sequent $\Gamma \mapsto \Delta$, if $\Gamma \mapsto \Delta$ is provable in G_1 then $\Gamma \mapsto \Delta$ is valid; and conversely, if $\Gamma \mapsto \Delta$ is valid then $\Gamma \mapsto \Delta$ is provable in G_1.

Formally, the validity of sequent $\Gamma \Rightarrow \Delta$ is defined as follows:

$\models_{G^1} \Gamma \Rightarrow \Delta$ if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 1$.

Correspondingly, sequent $\Gamma \not\Rightarrow \Delta$ (denoted by $\Gamma \mapsto \Delta$) being valid is defined as follows:

$\models_{G_1} \Gamma \Rightarrow \Delta$ if there is an assignment v such that $v \models \Gamma$ and $v \not\models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \not\models \Delta$ if for every $B \in \Delta, v(B) = 0$.

We consider other possible definitions of the validity and have the following four definitions:

- for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and

$$\begin{cases} v \models \Delta \text{ if for some B} \in \Delta, v(B) = 1; \\ v \models \Delta \text{ if for every B} \in \Delta, v(B) = 1; \\ v \models \Delta \text{ if for some B} \in \Delta, v(B) = 0; \\ v \models \Delta \text{ if for every B} \in \Delta, v(B) = 0; \end{cases}$$

- there is an assignment v such that $v \models \Gamma$ and $v \not\models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and

$$\begin{cases} v \not\models \Delta \text{ if for every B} \in \Delta, v(B) = 0; \\ v \not\models \Delta \text{ if for some B} \in \Delta, v(B) = 0; \\ v \not\models \Delta \text{ if for every B} \in \Delta, v(B) = 1; \\ v \not\models \Delta \text{ if for some B} \in \Delta, v(B) = 1. \end{cases}$$

Therefore, we have four Gentzen deduction systems G^1, G^2, G^3, G^4[2,9–11] and their dualities G_1, G_2, G_3, G_4, where

definition	system
$\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \exists B \in \Delta(v(B) = 1))$	$\mathbf{G}^1$
$\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \forall B \in \Delta(v(B) = 1))$	$\mathbf{G}^2$
$\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \exists B \in \Delta(v(B) = 0))$	$\mathbf{G}^3$
$\forall v(\forall A \in \Gamma(v(A) = 1) \Rightarrow \forall B \in \Delta(v(B) = 0))$	$\mathbf{G}^4$;

and

definition	system
$\exists v(\forall A \in \Gamma(v(A) = 1)\&\exists B \in \Delta(v(B) = 1))$	$\mathbf{G}_4$
$\exists v(\forall A \in \Gamma(v(A) = 1)\&\forall B \in \Delta(v(B) = 1))$	$\mathbf{G}_3$
$\exists v(\forall A \in \Gamma(v(A) = 1)\&\exists B \in \Delta(v(B) = 0))$	$\mathbf{G}_2$
$\exists v(\forall A \in \Gamma(v(A) = 1)\&\forall B \in \Delta(v(B) = 0))$	$\mathbf{G}_1$.

It will be proved that:[6,8,12–15]

(1) G^1, G^3 are monotonic in both Γ and Δ;
(2) G^2, G^4 are monotonic in Γ and nonmonotonic in Δ;
(3) G_1, G_3 are nonmonotonic in both Γ and Δ;
(4) G_2, G_4 are nonmonotonic in Γ and monotonic in Δ.

This paper is organized as follows: the next section gives the basic definitions in the propositional logic; the third section gives the Gentzen deduction system G^1 for the traditional propositional logic and G_1 for the nonmonotonic propositional logic; the fourth section gives the Gentzen deduction systems G^2 and G_2 and proves that they are sound and complete; the fifth section gives sound and complete Gentzen deduction systems G^3 and G_3 and analyzes their monotonicity; the sixth section lists the sound and complete Gentzen deduction systems G^4 and G_4; and the last section concludes the whole paper with the table of monotonicity of all the systems.

2. The logical language of the propositional logic

The logical language of the propositional logic consists of the following symbols:

- propositional variables: $p_0, p_1, ...$;
- logical connectives: $\neg, \wedge, \vee$, and
- auxiliary symbols: $(,)$.

A string A of symbols is a formula if

$$A ::= p|\neg p|A_1 \wedge A_2|A_1 \vee A_2.$$

The semantics of the propositional logic is given by an assignment v, a function from the propositional variables to $\{0, 1\}$.

Given an assignment v, a formula A is true in v, denoted by $v \models A$, if

$$\begin{cases} v(p) = 1 & \text{if } A = p \\ v(p) = 0 & \text{if } A = \neg p \\ v \models A_1 \& v \models A_2 & \text{if } A = A_1 \wedge A_2 \\ v \models A_1 or v \models A_2 & \text{if } A = A_1 \vee A_2, \end{cases}$$

where $\sim, \&, or$ are symbols used in the meta-language, and correspondingly, $\neg, \wedge, \vee$ are the ones used in the language. Therefore, $v \not\models A_1$ can be represented as $\sim (v \models A_1)$.

A sequent δ is a pair (Γ, Δ), denoted by $\Gamma \Rightarrow \Delta$, where Γ, Δ are sets of formulas.

A literal l is a propositional variable or the negation of a propositional variable.

3. The propositional logic G^1

A sequent $\Gamma \Rightarrow \Delta$ is valid, denoted by $\models_{G^1} \Gamma \Rightarrow \Delta$, if for any assignment $v, v(A) = 1$ for every $A \in \Gamma$ implies $v(B) = 1$ for some $B \in \Delta$.

The Gentzen deduction system G^1 consists of the following axioms and deduction rules:

- Axioms:

$$(A_{\Rightarrow}) \frac{incon(\Gamma) \text{ or } incon(\Delta) \text{ or } \Gamma \cap \Delta \neq \emptyset}{\Gamma \Rightarrow \Delta}$$

where Γ, Δ are sets of literals.

- Deduction rules:

$$(_{\Rightarrow}\wedge_1^L) \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\wedge^R) \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(_{\Rightarrow}\wedge_2^L) \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}$$

$$(_{\Rightarrow}\vee^L) \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\vee_1^R) \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(_{\Rightarrow}\vee_2^R) \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

Theorem 3.1 (The soundness and completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$,

$$\vdash_{G^1} \Gamma \Rightarrow \Delta \text{ iff } \models_{G^1} \Gamma \Rightarrow \Delta$$

The propositional logic of G_1

Definition 3.2 A sequent $\Gamma \mapsto \Delta$ is valid, denoted by $\models_{G_1} \Gamma \mapsto \Delta$ if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for each $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for each $B \in \Delta, v(B) = 0$.

A sequent $\Gamma \mapsto \Delta$ is not valid if $\Gamma \mapsto \Delta$ is unsatisfiable, i.e., there is no assignment v such that $v \models \Gamma$ and $v \models \Delta$.

Lemma 3.3 Given two sets Γ, Δ of literals, $\models_{G_1} \Gamma \mapsto \Delta$ if and only if Γ and Δ are consistent, and $\Gamma \cap \Delta = \emptyset$.

The Gentzen deduction system G_1 consists of the following axioms and deduction rules:

- Axioms:

$$(A_{\mapsto}) \frac{con(\Gamma) \ \& \ con(\Delta) \ \& \ \Gamma \cap \Delta = \emptyset}{\Gamma \mapsto \Delta}$$

where Δ, Γ are sets of literals.

- Deduction rules:

$$(_{\mapsto}\wedge^L) \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (_{\mapsto}\wedge_1^R) \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(_{\mapsto}\wedge_2^R) \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(_{\mapsto}\vee_1^L) \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (_{\mapsto}\vee^R) \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(_{\mapsto}\vee_2^L) \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta}$$

Definition 3.4 A sequent $\Gamma \mapsto \Delta$ is provable, denoted by $\vdash_{G_1} \Gamma \mapsto \Delta$ if there is a sequence $\{\Gamma_1 \mapsto \Delta_1, ..., \Gamma_n \mapsto \Delta_n\}$ such that $\Gamma_n \mapsto \Delta_n = \Gamma \mapsto \Delta$, and for each $1 \leq i \leq n, \Gamma_i \mapsto \Delta_i$ is an axiom or is deduced from the previous sequents by one of the deduction rules.

Theorem 3.5 (The soundness and completeness theorem). For any sequent $\Gamma \mapsto \Delta$,

$$\vdash_{G_1} \Gamma \mapsto \Delta \text{ iff } \models_{G_1} \Gamma \mapsto \Delta$$

4. The propositional logic G^2

Definition 4.1 A sequent $\Gamma \Rightarrow \Delta$ is G^2-valid, denoted by $\models_{G^2} \Gamma \Rightarrow \Delta$ if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for each $B \in \Delta, v(B) = 1$.

Proposition 4.2 Let Γ, Δ be sets of literals. $\models_{G^2} \Gamma \Rightarrow \Delta$ if and only if

$$\Delta \subseteq \Gamma \text{ or } incon(\Gamma).$$

Proof. Assume that $\Delta \subseteq \Gamma$ or $incon(\Gamma)$. Then, $\models_{G^2} \Gamma \Rightarrow \Delta$.

Conversely, assume that $\Delta \not\subseteq \Gamma$ and $con(\Gamma)$. There is a literal $l \in \Delta - \Gamma$. Define an assignment v such that for any propositional variable p,

$$v(p) = \begin{cases} 1 & \text{if } p \in \Gamma \\ 0 & \text{if } \neg p \in \Gamma \\ 0 & \text{if } p = l \\ 1 & \text{if } p = \neg l \\ 0 & \text{otherwise.} \end{cases}$$

Then, $v \models \Gamma$ and $v \not\models \Delta$.

The Gentzen deduction system G^2 consists of the following axioms and deduction rules:

- Axioms:

$$(A_{\Rightarrow}) \frac{\Delta \subseteq \Gamma \text{ or } incon(\Gamma)}{\Gamma \Rightarrow \Delta},$$

where Δ, Γ are sets of literals.

- Deduction rules:

$$(_{\Rightarrow}\wedge_1^L) \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\wedge^R) \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(_{\Rightarrow}\wedge_2^L) \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta}$$

$$(_{\Rightarrow}\vee^L) \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\vee_1^R) \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

$$(_{\Rightarrow}\vee_2^R) \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

Theorem 4.3 (The soundness theorem). For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{G^2} \Gamma \Rightarrow \Delta$ then $\models_{G^2} \Gamma \Rightarrow \Delta$.

Proof. We prove that each axiom is valid and each deduction rule preserves the validity.

To verify the validity of the axiom, by Proposition 4.2, the axiom is valid.

To verify that $(_\Rightarrow\wedge_i^L)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma, A_i \text{ implies } v \models \Delta$$

For any assignment v, assume that $v \models \Gamma, A_1 \wedge A_2$. Then, $v \models \Gamma, A_i$, and by the induction assumption, $v \models \Delta$.

To verify that $(_\Rightarrow\wedge^R)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma \text{ implies } v \models B_1, \Delta$$
$$v \models \Gamma \text{ implies } v \models B_2, \Delta$$

For any assignment v, assume that $v \models \Gamma$. By the induction assumption, $v \models B_1, \Delta$ and $v \models B_2, \Delta$. Hence, $v \models B_1 \wedge B_2, \Delta$.

To verify that $(_\Rightarrow\vee^L)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma, A_1 \text{ implies } v \models \Delta,$$
$$v \models \Gamma, A_2 \text{ implies } v \models \Delta$$

For any assignment v, assume that $v \models \Gamma, A_1 \vee A_2$. Then, either $v \models \Gamma, A_1$ or $v \models \Gamma, A_2$, and by the induction assumption, either case implies $v \models \Delta$.

To verify that $(_\Rightarrow\vee_i^R)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma \text{ implies } v \models B_i, \Delta$$

For any assignment v, assume that $v \models \Gamma$. By the induction assumption, $v \models B_i, \Delta$. Hence, $v \models B_1 \vee B_2, \Delta$.

4.1 The completeness theorem of G^2

Theorem 4.4 (The completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$, if $\models_{G^2} \Gamma \Rightarrow \Delta$ then $\vdash_{G^2} \Gamma \Rightarrow \Delta$.

Proof. Given a sequent $\Gamma \Rightarrow \Delta$, we construct a tree T as follows:

- the root of T is $\Gamma \Rightarrow \Delta$;
- if for each sequent $\Gamma' \Rightarrow \Delta'$ at a node, Γ', Δ' are sets of literals then the node is a leaf; and
- if a sequent $\Gamma' \Rightarrow \Delta'$ at a nonleaf node of T is not an axiom then the node has the direct child nodes

$$\begin{cases} \begin{bmatrix} \Gamma_1, A_1 \Rightarrow \Delta_1 \\ \Gamma_1, A_2 \Rightarrow \Delta_1 \end{bmatrix} & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1, A_1 \wedge A_2 \Rightarrow \Delta_1 \\ \begin{cases} \Gamma_1 \Rightarrow B_1, \Delta_1 \\ \Gamma_1 \Rightarrow B_2, \Delta_1 \end{cases} & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1 \Rightarrow B_1 \wedge B_2, \Delta_1 \\ \begin{cases} \Gamma_1, A_1 \Rightarrow \Delta_1 \\ \Gamma_1, A_2 \Rightarrow \Delta_1 \end{cases} & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1, A_1 \vee A_2 \Rightarrow \Delta_1 \\ \begin{bmatrix} \Gamma_1 \Rightarrow B_1, \Delta_1 \\ \Gamma_1 \Rightarrow B_2, \Delta_1 \end{bmatrix} & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1 \Rightarrow B_1 \vee B_2, \Delta_1, \end{cases}$$

where $\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix}$ represents that δ_1, δ_2 are at a same child node; and $\begin{cases}\delta_1 \\ \delta_2\end{cases}$ represents that δ_1, δ_2 are at different direct child nodes;

Theorem 4.5 If there is a branch $\xi \subseteq T$ such that the leaf of ξ is not an axiom in G^2 then there is an assignment v such that $v \not\models_{G^2} \Gamma \Rightarrow \Delta$.

Proof. Assume that the leaf of ξ is not an axiom in G^2, and let the leaf be $\Gamma' \Rightarrow \Delta'$, Then, By Proposition 4.2, there is an assignment v such that $v \not\models_{G^2} \Gamma' \Rightarrow \Delta'$.

We shall prove that for each node $\Gamma_1 \Rightarrow \Delta_1$ of $\xi, v \not\models_{G^2} \Gamma_1 \Rightarrow \Delta_1$. There are the following cases for $\Gamma_1 \Rightarrow \Delta_1$.

Case 1. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2, A_1, A_2 \Rightarrow \Delta_2 \in \xi$ is a direct child nodes of $\Gamma_2, A_1 \wedge A_2 \Rightarrow \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^2} \Gamma_2, A_1, A_2 \mapsto \Delta_2$$

and we have that $v \not\models_{G^2} \Gamma_2, A_1 \wedge A_2 \mapsto \Delta_2$.

Case 2. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2 \Rightarrow B_i, \Delta_2 \in \xi$ is a direct child node of $\Gamma_2 \Rightarrow B_1 \wedge B_2, \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^2} \Gamma_2 \Rightarrow B_i, \Delta_2$$

and we have that $v \not\models_{G^2} \Gamma_2 \Rightarrow B_1 \wedge B_2, \Delta_2$.

Case 3. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2, A_i \Rightarrow \Delta_2 \in \xi$ is a direct child node of $\Gamma_2, A_1 \vee A_2 \Rightarrow \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^2} \Gamma_2, A_i \Rightarrow \Delta_2$$

and we have that $v \not\models_{G^2} \Gamma_2, A_1 \vee A_2 \Rightarrow \Delta_2$.

Case 4. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2 \Rightarrow B_1, B_2, \Delta_2 \in \xi$ is a direct child node of $\Gamma_2 \Rightarrow B_1 \vee B_2, \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^2} \Gamma_2 \Rightarrow B_1, B_2, \Delta_2$$

that is,

$$v \not\models_{G^2} \Gamma_2 \Rightarrow B_1, \Delta_2,$$

$$v \not\models_{G^2} \Gamma_2 \Rightarrow B_2, \Delta_2$$

and we have that $v \not\models_{G^2} \Gamma_2 \Rightarrow B_1 \vee B_2 \Delta_2$.

Theorem 4.6 If each leaf $\Gamma' \Rightarrow \Delta'$ of T is an axiom in G^2 then T is a proof tree of $\Gamma \Rightarrow \Delta$ in G^2.

Proof. The theorem follows directly from the definition of T.

4.2 The propositional logic G_2

Definition 4.7 A sequent $\Gamma \mapsto \Delta$ is G_2-valid, denoted by $\models_{G_2} \Gamma \mapsto \Delta$ if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 0$.

Proposition 4.8 Let Γ, Δ be sets of literals. $\models_{G_2} \Gamma \mapsto \Delta$ if and only if

$$\Delta \not\subseteq \Gamma \ \& \ con(\Gamma).$$

The Gentzen deduction system G_2 consists of the following axioms and deduction rules:

- Axioms:

$$(A_{\mapsto}) \ \frac{\Delta \not\subseteq \Gamma \ \& \ con(\Gamma)}{\Gamma \mapsto \Delta},$$

  where Δ, Γ are sets of literals.
- Deduction rules:

$$(\mapsto\wedge^L) \ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (\mapsto\wedge_1^R) \ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\wedge_2^R) \ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\vee_1^L) \ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee^R) \ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\vee_2^L) \ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta}$$

Theorem 4.9 (The soundness theorem). For any sequent $\Gamma \mapsto \Delta$, if $\vdash_{G_2} \Gamma \mapsto \Delta$ then $\models_{G_2} \Gamma \mapsto \Delta$.

Proof. We prove that each axiom is valid and each deduction rule preserves the validity.

To verify the validity of the axiom, by Proposition 4.8, the axiom is valid.

To verify that $(\mapsto\wedge^L)$ preserves the validity, assume that there is an assignment v such that

$$v \models \Gamma, A_1 \& v \models \Delta,$$
$$v \models \Gamma, A_2 \& v \models \Delta.$$

For this assignment v, $v \models \Gamma, A_1 \wedge A_2$ and $v \models \Delta$.

To verify that $(\mapsto\wedge_i^R)$ preserves the validity, assume that there is an assignment v such that

$$v \models \Gamma \& v \models B_i, \Delta.$$

For this assignment v, $v \models \Gamma$ and $v \models B_1 \wedge B_2, \Delta$.

To verify that $(\mapsto\vee_i^L)$ preserves the validity, assume that there is an assignment v such that

$$v \models \Gamma, A_i \& v \models \Delta.$$

For this assignment v, $v \models \Gamma, A_1 \vee A_2$ and $v \models \Delta$.

To verify that $(\mapsto\vee^R)$ preserves the validity, assume that there is an assignment v such that

$$v \models \Gamma \& v \models B_1, \Delta,$$
$$v \models \Gamma \& v \models B_2, \Delta.$$

For this assignment v, $v \models \Gamma$ and $v \models B_1 \vee B_2, \Delta$.

4.3 The completeness theorem of G_2

Theorem 4.10 (The completeness theorem). For any sequent $\Gamma \mapsto \Delta$, if $\models_{G_2} \Gamma \mapsto \Delta$ then $\vdash_{G_2} \Gamma \mapsto \Delta$.

Proof. Given a sequent $\Gamma \mapsto \Delta$, we construct a tree T as follows:

- the root of T is $\Gamma \mapsto \Delta$;
- if for each sequent $\Gamma' \mapsto \Delta'$ at a node, Γ', Δ' are sets of literals then the node is a leaf; and
- if a sequent $\Gamma' \mapsto \Delta'$ at a nonleaf node of T is not an axiom then the node has the direct child nodes

$$\begin{cases} \left[\begin{array}{l} \Gamma_1, A_1 \mapsto \Delta_1 \\ \Gamma_1, A_2 \mapsto \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1, A_1 \wedge A_2 \mapsto \Delta_1 \\ \left\{\begin{array}{l} \Gamma_1 \mapsto B_1, \Delta_1 \\ \Gamma_1 \mapsto B_2, \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1 \mapsto B_1 \wedge B_2, \Delta_1 \\ \left\{\begin{array}{l} \Gamma_1, A_1 \mapsto \Delta_1 \\ \Gamma_1, A_2 \mapsto \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1, A_1 \vee A_2 \mapsto \Delta_1 \\ \left[\begin{array}{l} \Gamma_1 \mapsto B_1, \Delta_1 \\ \Gamma_1 \mapsto B_2, \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1 \mapsto B_1 \vee B_2, \Delta_1 \end{cases}$$

Theorem 4.11 If there is a branch $\xi \subseteq T$ such that the leaf of ξ is a precondition of the axiom in G_2 then $\vdash_{G_2} \Gamma \mapsto \Delta$.

Proof. Assume that the leaf of ξ is a precondition of the axiom in G_2, by $(\mapsto A)$, $\vdash_{G_2} \Gamma' \mapsto \Delta'$.

We shall prove that for each node $\Gamma_1 \mapsto \Delta_1$ of ξ, $\vdash_{G_2} \Gamma_1 \mapsto \Delta_1$. There are the following cases for $\Gamma_1 \mapsto \Delta_1$.

Case 1. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2, A_1, A_2 \mapsto \Delta_2 \in \xi$ is a direct child node of $\Gamma_2, A_1 \wedge A_2 \mapsto \Delta_2 \in \xi$. By the induction assumption,

$$\vdash_{G_2} \Gamma_2, A_1 \mapsto \Delta_2,$$

$$\vdash_{G_2} \Gamma_2, A_2 \mapsto \Delta_2,$$

and by $(\mapsto\wedge^L)$, we have that $\vdash_{G_2} \Gamma_2, A_1 \wedge A_2 \mapsto \Delta_2$.

Case 2. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2 \mapsto B_i, \Delta_2 \in \xi$ is a direct child node of $\Gamma_2 \mapsto B_1 \wedge B_2, \Delta_2 \in \xi$. By the induction assumption,

$$\vdash_{G_2} \Gamma_2 \mapsto B_i, \Delta_2,$$

and by $(\mapsto\wedge_i^R)$, we have that $\vdash_{G_2} \Gamma_2 \mapsto B_1 \wedge B_2, \Delta_2$.

Case 3. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2, A_i \mapsto \Delta_2 \in \xi$ is a direct child node of $\Gamma_2, A_1 \vee A_2 \mapsto \Delta_2 \in \xi$. By the induction assumption,

$$\vdash_{G_2} \Gamma_2, A_i \mapsto \Delta_2,$$

and by $(\mapsto\vee_i^L)$, we have that $\vdash_{G_2} \Gamma_2, A_1 \vee A_2 \mapsto \Delta_2$.

Case 4. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2 \mapsto B_1, B_2, \Delta_2 \in \xi$ is a direct child node of $\Gamma_2 \mapsto B_1 \vee B_2, \Delta_2 \in \xi$. By the induction assumption,

$$\vdash_{G_2} \Gamma_2 \mapsto B_1, B_2, \Delta_2,$$

that is,

$$\vdash_{G_2} \Gamma_2 \mapsto B_1, \Delta_2,$$

$$\vdash_{G_2} \Gamma_2 \mapsto B_2, \Delta_2,$$

and by $(\mapsto\vee^R)$, we have that $\vdash_{G_2} \Gamma_2 \mapsto B_1 \vee B_2\Delta_2$.

Theorem 4.12 If each leaf $\Gamma' \mapsto \Delta'$ of T is not an axiom in G_2 then T is a proof tree of $\Gamma \Rightarrow \Delta$ in G^2.

Proof. The theorem follows directly from the definition of T.

5. THE PROPOSITIONAL LOGIC G^3

Given a sequent $\Gamma \Rightarrow \Delta$, we say that v satisfies $\Gamma \Rightarrow \Delta$, denoted by $v \models_{G^3} \Gamma \Rightarrow \Delta$, if 1) that for each formula $A \in \Gamma, v(A) = 1$ implies 2) that for some formula $A \in \Delta, v(A) = 0$.

A sequent $\Gamma \Rightarrow \Delta$ is valid, denoted by $\models_{G^3} \Gamma \Rightarrow \Delta$, if for any assignment v, $v \models_{G^3} \Gamma \Rightarrow \Delta$.

Proposition 5.1 Let Γ, Δ be sets of literals. $\models_{G^3} \Gamma \Rightarrow \Delta$ if and only if

$$incon(\Gamma) \text{ or } incon(\neg\Delta) \text{ or } \Gamma \cap \neg\Delta \neq \emptyset.$$

Proof. $\models_{G^3} \Gamma \Rightarrow \Delta$ iff for any assignment v, 1) that for each formula $A \in \Gamma, v(A) = 1$ implies 2) that for some formula $A \in \Delta, v(A) = 0$; iff for any assignment v, 1) that for each formula $A \in \Gamma, v(A) = 1$ implies 2) that for some formula $A \in \Delta, v(\neg A) = 1$; iff $\models_{G_1} \Gamma \Rightarrow \neg\Delta$, iff

$$incon(\Gamma) \text{ or } incon(\neg\Delta) \text{ or } \Gamma \cap \neg\Delta \neq \emptyset,$$

where $\neg\Delta = \{\neg B : B \in \Delta\}$.

Proposition 5.2 Let Γ, Δ be sets of literals. $incon(\Gamma)$ or $incon(\neg\Delta)$ or $\Gamma \cap \neg\Delta \neq \emptyset$ if and only if $incon(\Gamma \cup \Delta)$.

The Gentzen deduction system G^3 contains the following axioms and deduction rules:

- Axioms:

$$\frac{incon(\Gamma) \text{ or } incon(\neg\Delta) \text{ or } \Gamma \cap \neg\Delta \neq \emptyset}{\Gamma \Rightarrow \Delta},$$

  where Γ, Δ are sets of literals.
- The deduction rules for connectives:

$$(\Rightarrow\wedge_1^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge_1^R)\ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\wedge_2^L)\ \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge_2^R)\ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(\Rightarrow\vee^L)\ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (\Rightarrow\vee^R)\ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

Definition 5.3 $\vdash_{G^3} \Gamma \Rightarrow \Delta$ if there is a sequence $\{\Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_n \Rightarrow \Delta_n$ such that $\Gamma_n \Rightarrow \Delta_n = \Gamma \Rightarrow \Delta$, and for each $1 \leq i \leq n, \Gamma_i \Rightarrow \Delta_i$ is an axiom or is deduced from the previous sequents by one of the deduction rules in G^3.

Theorem 5.4 (The soundness theorem). For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{G^3} \Gamma \Rightarrow \Delta$ then $\models_{G^3} \Gamma \Rightarrow \Delta$.

Proof. We prove that each axiom is valid and each deduction rule preserves the validity.

To verify the validity of the axiom, by Proposition 5.1, the axiom is valid.

To verify that $(\Rightarrow\wedge^{R_1})$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma, A_1 \text{ implies } v \models \Delta.$$

For any assignment v, assume that $v \models \Gamma, A_1 \wedge A_2$. Then, $v \models \Gamma, A_1$. By the induction assumption, $v \models \Delta$.

To verify that $(\Rightarrow\wedge^R)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma \text{ implies } v \models B_1, \Delta.$$

For any assignment v, assume that $v \models \Gamma$. By the induction assumption, $v \models B_1, \Delta$. If $v \models \Delta$ then $v \models B_1 \wedge B_2, \Delta$; otherwise, $v \models B_1$ and so $v \models B_1 \wedge B_2$, i.e., $v \models B_1 \wedge B_2, \Delta$.

To verify that $(\vee^L)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma, A_1 \text{ implies } v \models \Delta$$

$$v \models \Gamma, A_2 \text{ implies } v \models \Delta.$$

For any assignment v, assume that $v \models \Gamma, A_1 \vee A_2$. If $v \models A_1$ then $v \models \Gamma, A_1$, and by the induction assumption, $v \models \Delta$; and if $v \models A_2$ then, $v \models \Gamma, A_1$, and by the induction assumption, $v \models \Delta$.

To verify that $(\Rightarrow\vee^R)$ preserves the validity, assume that for any assignment v,

$$v \models \Gamma \text{ implies } v \models B_1, \Delta$$

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$$v \models \Gamma \text{ implies } v \models B_2, \Delta.$$

For any assignment v, assume that $v \models \Gamma$. By the induction assumption, $v \models B_1, \Delta$ and $v \models B_1, \Delta$. If $v \models \Delta$ then $v \models B_1 \vee B_2, \Delta$; otherwise, $v \models B_1; v \models B_2$, and so $v \models B_1 \vee B_2, \Delta$.

5.1 The completeness theorem of G^3

Theorem 5.5 (The completeness theorem). For any sequent $\Gamma \Rightarrow \Delta$, if $\models_{G^3} \Gamma \Rightarrow \Delta$ then $\vdash_{G^3} \Gamma \Rightarrow \Delta$.

Proof. Given a sequent $\Gamma \Rightarrow \Delta$, we construct a tree T as follows:

- the root of T is $\Gamma \Rightarrow \Delta$;
- if for each sequent $\Gamma' \Rightarrow \Delta'$ at a node, Γ', Δ' are sets of literals then the node is a leaf; and
- if a sequent $\Gamma' \Rightarrow \Delta'$ at a nonleaf node of T is not an axiom then the node has the direct child nodes

$$\left\{\begin{array}{ll} \left[\begin{array}{l} \Gamma_1, A_1 \Rightarrow \Delta_1 \\ \Gamma_1, A_2 \Rightarrow \Delta_1 \end{array}\right. & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1, A_1 \wedge A_2 \Rightarrow \Delta_1 \\ \left[\begin{array}{l} \Gamma_1 \Rightarrow B_1, \Delta_1 \\ \Gamma_1 \Rightarrow B_2, \Delta_1 \end{array}\right. & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1 \Rightarrow B_1 \wedge B_2, \Delta_1 \\ \left\{\begin{array}{l} \Gamma_1, A_1 \Rightarrow \Delta_1 \\ \Gamma_1, A_2 \Rightarrow \Delta_1 \end{array}\right. & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1, A_1 \vee A_2 \Rightarrow \Delta_1 \\ \left\{\begin{array}{l} \Gamma_1 \Rightarrow B_1, \Delta_1 \\ \Gamma_1 \Rightarrow B_2, \Delta_1 \end{array}\right. & \text{if } \Gamma' \Rightarrow \Delta' = \Gamma_1 \Rightarrow B_1 \vee B_2, \Delta_1, \end{array}\right.$$

Theorem 5.6 If there is a branch $\xi \subseteq T$ such that the leaf of ξ is not an axiom in G^3 then there is an assignment v such that $v \not\models_{G^3} \Gamma \Rightarrow \Delta$.

Proof. Assume that the leaf of ξ is not an axiom in G^3, and let the leaf be $\Gamma' \Rightarrow \Delta'$, Then, By Proposition 5.1, there is an assignment v such that $v \not\models_{G^3} \Gamma' \Rightarrow \Delta'$.

We shall prove that for each node $\Gamma_1 \Rightarrow \Delta_1$ of $\xi, v \not\models_{G^3} \Gamma_1 \Rightarrow \Delta_1$. There are the following cases for $\Gamma_1 \Rightarrow \Delta_1$.

Case 1. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2, A_1, A_2 \Rightarrow \Delta_2 \in \xi$ is a direct child node of $\Gamma_2, A_1 \wedge A_2 \Rightarrow \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^3} \Gamma_2, A_1, A_2 \Rightarrow \Delta_2,$$

and we have that $v \not\models_{G^3} \Gamma_2, A_1 \wedge A_2 \Rightarrow \Delta_2$.

Case 2. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2 \Rightarrow B_1, B_2, \Delta_2 \in \xi$ is a direct child node of $\Gamma_2 \Rightarrow B_1 \wedge B_2, \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^3} \Gamma_2 \Rightarrow B_1, B_2, \Delta_2,$$

and we have that $v \not\models_{G^3} \Gamma_2 \Rightarrow B_1 \wedge B_2, \Delta_2$.

Case 3. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2, A_i \Rightarrow \Delta_2 \in \xi$ is a direct child node of $\Gamma_2, A_1 \vee A_2 \Rightarrow \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^3} \Gamma_2, A_i \Rightarrow \Delta_2,$$

and we have that $v \not\models_{G^3} \Gamma_2, A_1 \vee A_2 \Rightarrow \Delta_2$.

Case 4. $\Gamma_1 \Rightarrow \Delta_1 = \Gamma_2 \Rightarrow B_i, \Delta_2 \in \xi$ is a direct child node of $\Gamma_2 \Rightarrow B_1 \vee B_2, \Delta_2 \in \xi$. By the induction assumption,

$$v \not\models_{G^3} \Gamma_2 \Rightarrow B_i, \Delta_2,$$

and we have that $v \not\models_{G^3} \Gamma_2 \Rightarrow B_1 \vee B_2 \Delta_2$.

Theorem 5.7 If each leaf $\Gamma' \Rightarrow \Delta'$ of T is an axiom in G^3 then T is a proof tree of $\Gamma \Rightarrow \Delta$ in G^3.

Proof. The theorem follows directly from the definition of T.

5.2 The propositional logic G_3

Definition 5.8 A sequent $\Gamma \mapsto \Delta$ is valid in G_3, denoted by $\models_{G_3} \Gamma \mapsto \Delta$ if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for each $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for each $B \in \Delta, v(B) = 1$.

A sequent $\Gamma \mapsto \Delta$ is not valid in G_3 if $\Gamma \mapsto \Delta$ is unsatisfiable in G_3, i.e., there is no assignment v such that $v \models \Gamma$ and $v \models \Delta$, equivalently, for any assignment $v, v \models \Gamma$ implies $v \models \Delta$.

Lemma 5.9 Given two sets Γ, Δ of literals, $\models_{G_3} \Gamma \mapsto \Delta$ if and only if Γ and $\neg\Delta$ are consistent, and $\Gamma \cap \neg\Delta = \emptyset$, equivalently, $con(\Gamma \cup \Delta)$.

The Gentzen deduction system G_3 consists of the following axioms and deduction rules:

- Axioms:

$$(A^{\text{'}} \rightarrow) \ \frac{con(\Gamma) \ \& \ con(\neg\Delta) \ \& \ \Gamma \cap \neg\Delta = \emptyset}{\Gamma \mapsto \Delta},$$

  where Δ, Γ are sets of literals.

- Deduction rules:

$$(\mapsto\wedge^L) \ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (\mapsto\wedge^R) \ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\vee_1^L) \ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee_1^R) \ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\vee_2^L) \ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee_2^R) \ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

Definition 5.10 A sequent $\Gamma \mapsto \Delta$ is provable, denoted by $\vdash_{G_3} \Gamma \mapsto \Delta$ if there is a sequence $\{\Gamma_1 \mapsto \Delta_1, ..., \Gamma_n \mapsto \Delta_n\}$ such that $\Gamma_n \mapsto \Delta_n = \Gamma \mapsto \Delta$, and for each $1 \leq i \leq n, \Gamma_i \mapsto \Delta_i$ is an axiom or is deduced from the previous sequents by one of the deduction rules.

5.3 The soundness and completeness theorem of G_3

Theorem 5.11 (The soundness theorem). For any sequent $\Gamma \mapsto \Delta$,

$$\vdash_{G_3} \Gamma \mapsto \Delta \text{ implies } \models_{G_3} \Gamma \mapsto \Delta.$$

Proof. We prove that each axiom is valid and each deduction rule preserves the validity.

To verify the validity of the axiom, assume that $con(\Gamma)$, $con(\neg\Delta)$ and $\Gamma \cap \neg\Delta = \emptyset$. By Lemma 5.2.8, there is an assignment v such that $v \models_{G_3} \Gamma \mapsto \Delta$.

To verify that $(_{\mapsto}\wedge^L)$ preserves the validity, assume that there is an assignment v such that

$$v(\Gamma, A_1) = 1 \& v \models \Delta,$$
$$v(\Gamma, A_2) = 1 \& v \models \Delta.$$

For this very assignment $v, v(\Gamma, A_1 \wedge A_2) = 1$ and $v \models \Delta$.

To verify that $(_{\mapsto}\wedge^R)$ preserves the validity, assume that there is an assignment v such that

$$v \models \Gamma \& v(\Delta, B_1) = 1,$$
$$v \models \Gamma \& v(\Delta, B_2) = 1.$$

For this very assignment $v, v \models \Gamma$ and $v(\Delta, B_1 \wedge B_2) = 1$.

To verify that $(_{\mapsto}\vee_1^L)$ preserves the validity, assume that there is an assignment v such that $v(\Gamma, A_1) = 1$ and $v \models \Delta$. For this very assignment $v, v(\Gamma, A_1 \vee A_2) = 1$ and $v \models \Delta$.

To verify that $(_{\mapsto}\vee_1^R)$ preserves the validity, assume that there is an assignment v such that $v \models \Gamma$ and $v(\Delta, B_1) = 1$. For this very assignment $v, v \models \Gamma$ and $v(\Delta, B_1 \vee B_2) = 1$.

Theorem 5.12 (The completeness theorem). For any sequent $\Gamma \mapsto \Delta$,

$$\models_{G_3} \Gamma \mapsto \Delta \text{ implies } \vdash_{G_3} \Gamma \mapsto \Delta.$$

Proof. Given a sequent $\Gamma \mapsto \Delta$, we construct a tree T as follows:

- the root of T is $\Gamma \mapsto \Delta$;
- if for each sequent $\Gamma' \mapsto \Delta'$ at a node, Γ', Δ' are sets of literals then the node is a leaf; and
- if a sequent $\Gamma' \mapsto \Delta'$ at a nonleaf node of T is not an axiom then the node has the direct child nodes

$$\begin{cases} \left[\begin{array}{l} \Gamma_1, A_1 \mapsto \Delta_1 \\ \Gamma_1, A_2 \mapsto \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1, A_1 \wedge A_2 \mapsto \Delta_1 \\ \left[\begin{array}{l} \Gamma_1 \mapsto B_1, \Delta_1 \\ \Gamma_1 \mapsto B_2, \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1 \mapsto B_1 \wedge B_2, \Delta_1 \\ \left\{\begin{array}{l} \Gamma_1, A_1 \mapsto \Delta_1 \\ \Gamma_1, A_2 \mapsto \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1, A_1 \vee A_2 \mapsto \Delta_1 \\ \left\{\begin{array}{l} \Gamma_1 \mapsto B_1, \Delta_1 \\ \Gamma_1 \mapsto B_2, \Delta_1 \end{array}\right. & \text{if } \Gamma' \mapsto \Delta' = \Gamma_1 \mapsto B_1 \vee B_2, \Delta_1 \end{cases}$$

Theorem 5.13 If there is a branch $\xi \subseteq T$ such that the leaf of ξ is an axiom in G_3 then $\vdash_{G_3} \Gamma \mapsto \Delta$.

Proof. Assume that the leaf of ξ is an axiom in G_3. Let the leaf be $\Gamma' \mapsto \Delta'$. Then, $con(\Gamma'), con(\neg\Delta')$ and $\Gamma' \cap \neg\Delta' = \emptyset$, and by $(A), \vdash_{G_3} \Gamma' \Rightarrow \Delta'$.

We shall prove that for each node $\Gamma_1 \mapsto \Delta_1$ of $\xi, \vdash_{G_3} \Gamma_1 \Rightarrow \Delta_1$. There are the following cases for $\Gamma_1 \mapsto \Delta_1$.

Case 1. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2, A_1 \wedge A_2 \mapsto \Delta_2 \in \xi$. Then, $\Gamma_1 \mapsto \Delta_1$ has a direct child node $\Gamma_2, A_1, A_2 \mapsto \Delta_2$ (that is, $\Gamma_2, A_1 \mapsto \Delta_2$ $\Gamma_2, A_2 \mapsto \Delta_2$). By the assumption, we have

$$\vdash_{G_3} \Gamma_2, A_1 \mapsto \Delta_2,$$
$$\vdash_{G_3} \Gamma_2, A_2 \mapsto \Delta_2,$$

and by $(_{\mapsto}\wedge^L)$ in $G_3, \vdash_{G_3} \Gamma_2, A_1 \wedge A_2 \mapsto \Delta_2$.

Case 2. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2, A_1 \vee A_2 \mapsto \Delta_2 \in \xi$. Then, $\Gamma_1 \mapsto \Delta_1$ has two direct child node $\Gamma_2, A_1 \mapsto \Delta_2$ and $\Gamma_2, A_2 \mapsto \Delta_2$. There is an $i \in \{1, 2\}$ such that $\Gamma_2, A_i \mapsto \Delta_2 \in \xi$. By the induction assumption, we have

$$\vdash_{G_3} \Gamma_2, A_1 \mapsto \Delta_2,$$

and by $(_{\mapsto}\vee^L)$ in $G_3, \vdash_{G_3} \Gamma_2, A_1 \vee A_2 \Rightarrow \Delta_2$.

Case 3. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2 \mapsto B_1 \wedge B_2, \Delta_2 \in \xi$. Then, $\Gamma_1 \mapsto \Delta_1$ has a direct child node $\Gamma_2 \mapsto B_1, B_2, \Delta_2$ (that is, $\Gamma_2 \mapsto B_1, \Delta_2$ $\Gamma_2 \mapsto B_2, \Delta_2$). By the induction assumption,

$$\vdash_{G_3} \Gamma_2 \mapsto B_1, \Delta_2,$$
$$\vdash_{G_3} \Gamma_2 \mapsto B_2, \Delta_2,$$

and by $(_{\mapsto}\wedge^R)$ in $G_3, \vdash_{G_3} \Gamma_2 \mapsto B_1 \wedge B_2, \Delta_2$.

Case 4. $\Gamma_1 \mapsto \Delta_1 = \Gamma_2 \mapsto B_1 \vee B_2, \Delta_2 \in \xi$. Then, $\Gamma_1 \mapsto \Delta_1$ has two direct child nodes $\Gamma_2 \mapsto B_1, \Delta_2$ and $\Gamma_2 \mapsto B_2, \Delta_2$. There is an $i \in \{1, 2\}$ such that $\Gamma_2 \mapsto B_i, \Delta_2 \in \xi$. By the assumption, we have

$$\vdash_{G_3} \Gamma_2 \mapsto B_i, \Delta_2,$$

and by $(_{\mapsto}\vee_i^R)$ in $G_3, \vdash_{G_3} \Gamma_2 \mapsto B_1 \vee B_2, \Delta_2$.

Theorem 5.14 If each leaf $\Gamma' \mapsto \Delta'$ of T is not an axiom in G_3 then T is a proof tree of $\Gamma \Rightarrow \Delta$ in G^3.

Proof. Directly from the definition of T.

5.4 The nonmonotonicity of G_3

Theorem 5.15 (The monotonicity theorem). G^3 is monotonic in both Γ and Δ, that is, for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{G^3} \Gamma \mapsto \Delta \text{ implies } \vdash_{G^3} \Gamma' \mapsto \Delta;$$
$$\Delta \subseteq \Delta' \& \vdash_{G^3} \Gamma \mapsto \Delta \text{ implies } \vdash_{G^3} \Gamma \mapsto \Delta'.$$

Theorem 5.16 (The nonmonotonicity theorem). G_3 is nonmonotonic in both Γ and Δ, that is, for any formula sets

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Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \vdash_{G_3} \Gamma \mapsto \Delta \text{ may not imply } \vdash_{G_3} \Gamma' \mapsto \Delta;$$

$$\Delta \subseteq \Delta' \& \vdash_{G_3} \Gamma \mapsto \Delta \text{ may not imply } \vdash_{G_3} \Gamma \mapsto \Delta'.$$

Proof. We prove that the axiom is nonmonotonic and each deduction rule preserves the monotonicity.

Assume that $con(\Gamma), con(\neg\Delta)$ and $\Gamma \cap \neg\Delta = \emptyset$. There is a superset $\Gamma' \supseteq \Gamma$ such that $\Gamma' \cap \neg\Delta \neq \emptyset$; and there is a superset $\Delta' \supseteq \Delta$ such that $\Gamma \cap \neg\Delta' \neq \emptyset$. Hence, G_3 is nonmonotonic in both Γ and Δ.

To show that $(_{\mapsto}\wedge^R)$ preserves the monotonicity of Γ, assume that $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$ are monotonic with respect to Γ. By $(_{\mapsto}\wedge^R)$, from $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$, we infer $\Gamma \mapsto B_1 \wedge B_2, \Delta$. Then, for any $\Gamma' \supseteq \Gamma, \Gamma' \mapsto B_1, \Delta$ and $\Gamma' \mapsto B_2, \Delta$ follows by the assumptions; and by $(_{\mapsto}\wedge^R)$, from $\Gamma' \mapsto B_1, \Delta$ and $\Gamma' \mapsto B_2, \Delta$, we infer $\Gamma' \mapsto B_1 \wedge B_2, \Delta$. Hence, $\Gamma \mapsto B_1 \wedge B_2, \Delta$ implies $\Gamma' \mapsto B_1 \wedge B_2, \Delta$, that is, $\Gamma \mapsto B_1 \wedge B_2, \Delta$ is monotonic with respect to Γ.

To show that $(_{\mapsto}\wedge^R)$ preserves the nonmonotonicity of Γ, assume that $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$ are nonmonotonic with respect to Γ. By $(_{\mapsto}\wedge^R)$, from $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$, we infer $\Gamma \mapsto B_1 \wedge B_2, \Delta$. Then, for some $\Gamma' \supseteq \Gamma$,

$$\Gamma \mapsto A_1, \Delta \text{ may not imply } \Gamma', A_1 \not\mapsto \Delta;$$

$$\Gamma \mapsto A_2, \Delta \text{ may not imply } \Gamma', A_2 \not\mapsto \Delta;$$

and by $(_{\mapsto}\wedge^R), \Gamma \mapsto B_1 \wedge B_2, \Delta$ may not imply $\Gamma' \mapsto B_1 \wedge B_2, \Delta$, that is, $\Gamma \mapsto B_1 \wedge B_2, \Delta$ is nonmonotonic with respect to Γ.

To show that $(_{\mapsto}\wedge^R)$ preserves the monotonicity of Δ, assume that $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$ are monotonic with respect to Δ. By $(_{\mapsto}\wedge^R)$, from $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_1, \Delta$, we infer $\Gamma \mapsto B_1 \wedge B_2, \Delta$. Then, for any $\Delta' \supseteq \Delta, \Gamma \mapsto B_1, \Delta'$ and $\Gamma \mapsto B_2, \Delta$ follows; and by $(_{\mapsto}\wedge^R)$, from $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$, we infer $\Gamma \mapsto B_1 \wedge, \Delta'$. Hence, $\Gamma \mapsto B_1 \wedge B_2, \Delta$ implies $\Gamma \mapsto B_1 \wedge B_2, \Delta'$, that is, $\Gamma \mapsto B_1 \wedge B_2, \Delta$ is monotonic with respect to Δ.

To show that $(_{\mapsto}\wedge^R)$ preserves the nonmonotonicity of Δ, assume that $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$ are nonmonotonic with respect to Δ'. By $(_{\mapsto}\wedge^R)$, from $\Gamma \mapsto B_1, \Delta$ and $\Gamma \mapsto B_2, \Delta$, we infer $\Gamma \mapsto B_1 \wedge B_2, \Delta$. Then, for some $\Delta' \supseteq \Delta$,

$$\Gamma \mapsto B_1, \Delta \text{ may not imply } \Gamma \mapsto B_1, \Delta';$$

$$\Gamma \mapsto B_2, \Delta \text{ may not imply } \Gamma \mapsto B_2, \Delta';$$

and by $(_{\mapsto}\wedge^R), \Gamma \mapsto B_1 \wedge B_2, \Delta$ may not imply $\Gamma \mapsto B_1 \wedge B_2, \Delta'$, that is, $\Gamma \mapsto B_1 \wedge B_2, \Delta$ is nonmonotonic with respect to Δ.

Similar to show that other deduction rules preserve the monotonicity and nonmonotonicity with respect to Γ and Δ.

By the soundness and completeness theorem, we have that for any formula sets Γ, Γ', Δ and Δ',

$$\Gamma \subseteq \Gamma' \& \models_{G_3} \Gamma \mapsto \Delta \text{ may not imply } \models_{G_3} \Gamma' \mapsto \Delta,$$

$$\Delta \subseteq \Delta' \& \models_{G_3} \Gamma \mapsto \Delta \text{ may not imply } \models_{G_3} \Gamma \mapsto \Delta'.$$

6. The propositional logic G^4

Definition 6.1 A sequent $\Gamma \Rightarrow \Delta$ is G^4-valid, denoted by $\models_{G^4} \Gamma \Rightarrow \Delta$ if for any assignment $v, v \models \Gamma$ implies $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for each $B \in \Delta, v(B) = 0$.

Proposition 6.2 Let Γ, Δ be sets of literals. $\models_{G^4} \Gamma \Rightarrow \Delta$ if and only if

$$\neg\Delta \subseteq \Gamma \text{ or } incon(\Gamma).$$

Proof. Assume that $\neg\Delta \subseteq \Gamma or incon(\Gamma)$. Then, $\models_{G^4} \Gamma \Rightarrow \Delta$.

Conversely, assume that $\neg\Delta \not\subseteq \Gamma$ and $con(\Gamma)$. There is a literal $l \in \neg\Delta - \Gamma$. Define an assignment v such that for any propositional variable p,

$$v(p) = \begin{cases} 1 & \text{if } p \in \Gamma \\ 0 & \text{if } \neg p \in \Gamma \\ 0 & \text{if } p = l \\ 1 & \text{if } p = \neg l \\ 0 & \text{otherwise.} \end{cases}$$

Then, $v \models \Gamma$ and $v \not\models \Delta$.

The Gentzen deduction system G^4 consists of the following axioms and deduction rules:

- Axioms:

$$(A_{\Rightarrow}) \ \frac{\neg\Delta \not\subseteq \Gamma \ \& \ con(\Gamma)}{\Gamma \Rightarrow \neg\Delta},$$

  where Δ, Γ are sets of literals.

- Deduction rules:

$$(_{\Rightarrow}\wedge_1^L) \ \frac{\Gamma, A_1 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\wedge_1^R) \ \frac{\Gamma \Rightarrow B_1, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(_{\Rightarrow}\wedge_2^L) \ \frac{\Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\wedge_2^R) \ \frac{\Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \wedge B_2, \Delta}$$

$$(_{\Rightarrow}\vee^L) \ \frac{\Gamma, A_1 \Rightarrow \Delta \quad \Gamma, A_2 \Rightarrow \Delta}{\Gamma, A_1 \vee A_2 \Rightarrow \Delta} \qquad (_{\Rightarrow}\vee^R) \ \frac{\Gamma \Rightarrow B_1, \Delta \quad \Gamma \Rightarrow B_2, \Delta}{\Gamma \Rightarrow B_1 \vee B_2, \Delta}$$

Theorem 6.3 (The soundness and completeness theorem).

For any sequent $\Gamma \Rightarrow \Delta$,

$$\vdash_{G^4} \Gamma \Rightarrow \Delta \text{ iff } \models_{G^4} \Gamma \Rightarrow \Delta.$$

The propositional logic G_4

Definition 6.4 A sequent $\Gamma \mapsto \Delta$ is G_4-valid, denoted by $\models_{G_4} \Gamma \mapsto \Delta$ if there is an assignment v such that $v \models \Gamma$ and $v \models \Delta$, where $v \models \Gamma$ if for every $A \in \Gamma, v(A) = 1$; and $v \models \Delta$ if for some $B \in \Delta, v(B) = 1.5$.

Proposition 6.5 Let Γ, Δ be sets of literals. $\models_{G_4} \Gamma \mapsto \Delta$ if and only if

$$\neg\Delta \not\subseteq \Gamma \& con(\Gamma).$$

The Gentzen deduction system G_4 consists of the following axioms and deduction rules:

- Axioms:

$$(A_{\mapsto}) \ \frac{\neg\Delta \not\subseteq \Gamma \ \& \ con(\Gamma)}{\Gamma \mapsto \Delta},$$

  where Δ, Γ are sets of literals.
- Deduction rules:

$$(\mapsto\wedge^L) \ \frac{\Gamma, A_1 \mapsto \Delta \quad \Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \wedge A_2 \mapsto \Delta} \qquad (\mapsto\wedge^R) \ \frac{\Gamma \mapsto B_1, \Delta \quad \Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \wedge B_2, \Delta}$$

$$(\mapsto\vee_1^L) \ \frac{\Gamma, A_1 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee_1^R) \ \frac{\Gamma \mapsto B_1, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

$$(\mapsto\vee_2^L) \ \frac{\Gamma, A_2 \mapsto \Delta}{\Gamma, A_1 \vee A_2 \mapsto \Delta} \qquad (\mapsto\vee_2^R) \ \frac{\Gamma \mapsto B_2, \Delta}{\Gamma \mapsto B_1 \vee B_2, \Delta}$$

Theorem 6.6 (The soundness and completeness theorem). For any sequent $\Gamma \mapsto \Delta$,

$$\vdash_{G_4} \Gamma \mapsto \Delta \text{ iff } \models_{G_4} \Gamma \mapsto \Delta.$$

7. CONCLUSIONS

In this paper we proved that $G^1, G^2, G^3, G^4, G_1, G_2, G_3, G_4$ are sound and complete, and their monotonicity given in the following table:

system	precondition	mono Γ	mono Δ
$\mathbf{G}^1$	$\text{incon}(\Gamma) \vee \text{incon}(\Delta) \vee \Gamma \cap \Delta \neq \emptyset$	Y	Y
$\mathbf{G}_1$	$\text{con}(\Gamma) \wedge \text{con}(\Delta) \wedge \Gamma \cap \Delta = \emptyset$	N	N
$\mathbf{G}^2$	$\Delta \subseteq \Gamma \vee \text{incon}(\Gamma)$	Y	N
$\mathbf{G}_2$	$\Delta \not\subseteq \Gamma \wedge \text{con}(\Gamma)$	N	Y
$\mathbf{G}^3$	$\text{incon}(\Gamma) \vee \text{incon}(\neg\Delta) \vee \Gamma \cap \neg\Delta \neq \emptyset$	Y	Y
$\mathbf{G}_3$	$\text{con}(\Gamma) \wedge \text{con}(\neg\Delta) \wedge \Gamma \cap \neg\Delta = \emptyset$	N	N
$\mathbf{G}^4$	$\neg\Delta \subseteq \Gamma \vee \text{incon}(\Gamma)$	Y	N
$\mathbf{G}_4$	$\neg\Delta \not\subseteq \Gamma \wedge \text{con}(\Gamma)$	N	Y

where $\neg\Gamma = \{\neg A : A \in \Gamma\}$. Hence, $\neg\Gamma$ is consistent if and only if Γ is consistent.

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