ORIGINAL RESEARCH

Interval-valued process data monitoring and controlling

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Abstract

Statistical process control, a recognized technique for improving quality and productivity, has been widely employed throughout various industries. The conventional Shewhart control charts are applicable only when the collected sample data are real-valued data. For the purpose of controlling uncertain information when interval-valued data inevitably appear in the manufacturing or service processes, in this paper an interval-data analysis methodology is first applied. We construct Shewhart control charts whose control limits, consequently as interval numbers, are obtained by using the united extension principle, which is an effective method for dealing with closed interval data. Then, to identify the special causes of variation and alarm the requirement for corrective actions, we propose new rules for classifying current conditions of the manufacturing process based on an acceptability function of two interval numbers constructed from interval-valued sample data. Finally, the proposed methodologies are illustrated by practical examples to show their potential applications.

Key words

Interval-valued numbers, United extension principle, Acceptability function, Process control

1 Introduction

Over the fast development of new technologies, data with low resolution gathered from manufacturing processes, such as the synthesis and characterization of nano-composite processes or surface roughness consisting of the finer irregularities of the surface texture, thus recorded as interval values are commonly seen [1, 2]. There are several situations and examples for interval observations are inevitable existing in today's engineering processes. During earlier design phases of manufacturing processes, engineers may only know roughly in advance what the quality characteristics are looking for [3]; also, in the manufacturing period, timely and accurate numerical measurements of quality characteristics are sometime too costly to be obtained. Especially data gathered by human's subjective senses are rarely measured on an exact numerical scale [2, 4, 5-7]. A typical example for vague observations is the colors of the visible light spectrum usually recorded as an interval number due to insufficient resolution [8]. Moreover, measurements collected from color intensity of pictures, the sharpness or fineness of images, and indicators of an analog equipment are laborious and sometimes controversial to be exact [9, 10]. Even readings on digital measurement equipments are not precise numbers but certain spans since there are only finite numbers of decimals available [11].

In monitoring and controlling manufacturing processes, many researches have carried out that data collected from the key quality characteristic are in the form of qualitative variables, which by convention are called linguistic variables or categorical variables. Spanos and Chen [12] presented an example in which quality characteristics are measured the roughness of etched sidewalls, then trained operators classify wafers into particular categories such as "very rough", "rough", "smooth" and "very smooth". Fasulo et al. [13] investigated the surface quality of the thermoplastic olefin (TPO) nano-composites in an extrusion process, where surfaces quality is graded on a 5-point scale with 1 being the best and 5 being the worst based on visual inspections. Wang and Tsung [2] studied a Deep Reactive Ion Etching (DRIE) process, in which categorical observations were collected for the determination of process adjustment. Trochim and Donnelly [14] pointed out that all qualitative data can be coded quantitatively. Then these quantitative values can be manipulated to help decision-makers achieve greater insight into the meaning of the data and further examine specific hypotheses. Obviously, while assigning the qualitative variables to be meaningful numerical values, a certain degree of uncertainty called vagueness other than randomness is involved and thus yields coded data virtually interval-valued [15-17].

Statistical process control is a powerful collection of problem-solving tools that are useful in attaining the process stability and in improving the capability through the reduction of variability [18, 19]. A control chart is one of the major tools of the SPC that is commonly used to monitor and control the manufacturing process. The merits of the control charts lie in their ability to detect the process shifts and deviations and to indicate abnormal conditions [20]. Unfortunately, the conventional Shewhart control charts are applicable only when the collected sample data are real-valued data; that is, the manufacturing process is in control if the range r_i or the average $\overline{x}_i$ of the key quality characteristic is within the upper and lower control limits, and it is implied that the manufacturing process is out of control if the range r_i or the average $\overline{x}_i$ lies beyond the upper and lower control limits. In this case, the binary classification of “in control” or “out of control” is used to categorize the condition of the manufacturing process. For the interval-valued process data, this kind of two binary classifications might be too restrictive to make the right decision. While the interval-value process data are ubiquitous, an operative use for monitoring and controlling interval-valued quality data still has not been seriously treated.

Segupta and Pal [21] indicated interval numbers can be thought as the extension of real numbers as well as subsets of the real line $\mathbb{R}$. They convey the extent of tolerance that the key quality characteristic can possibly take. When a set of interval-valued sample data is collected for identifying if the special causes of variation exist or signaling if corrective actions are required, some key issues must be carried out to make control charts capable of being used. They are (1) the construction of the interval-valued upper and lower control limits; (2) the comparison of two interval numbers; (3) the categorization of the manufacturing process conditions. To fulfill these requirements, this paper is organized as follows. In Section 2, we briefly review the development of the equations for constructing the control limits on the Shewhart $\overline{X}$ and R charts. In Section 3, the definition of interval numbers and the united extension principle are introduced. In Section 4, we develop the $\overline{X}$ and R control charts with the interval-valued sample data. The interval upper and lower control limits are derived according to the united extension operations of interval analysis. In Section 5, a new method for comparing interval numbers so as to categorize the conditions of the manufacturing process is proposed Finally, the proposed methodologies are demonstrated by a practical example to show the potential application.

2 Control charts for real-valued data

We now briefly review the development of the equations for constructing the control limits on the $\overline{X}$ and R control charts. Suppose that X is the key quality characteristic, which has normal distribution $N(\mu,\sigma^2)$. Usually, the parameters μ and σ are numerically unknown. They need to be estimated by using the sample data taken from the process that is thought to be in control. These estimates should usually be obtained based on at least 20 to 25 samples.

Assume that we have collected the sample data $(X_{i1}, X_{i2}, \ldots, X_{in})$, $i = 1, 2, \ldots, m$, which consist of m subgroups and each subgroup contains n observations. The average of sample i is

$$\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \quad \text{for } i = 1, 2, \ldots, m \tag{1}$$

Then the best estimator of the true process average μ is the grand average

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m} \bar{X}_i \,. \tag{2}$$

On the other hand, in many applications of statistics to the quality engineering problems, it is convenient to estimate the standard deviation σ by the range method. The range of each sample is the difference between the largest and smallest observation. Let R_i be the range of the i-th sample, that is,

$$R_i = \max\{X_{i1}, X_{i2}, \ldots, X_{in}\} - \min\{X_{i1}, X_{i2}, \ldots, X_{in}\} \qquad \text{for } i = 1, 2, \ldots, m \,.$$

An unbiased estimator of the standard deviation σ of a normal distribution is $\hat{\sigma} = R_i / d_2$. The values of d_2 for various sample sizes $2 \le n \le 25$ are available in the textbooks or literature of quality control (e.g., see Montgomery [19]). Since there are m subgroups, the average range $\bar{R}$ is given by

$$\bar{R} = \frac{1}{m}\sum_{i=1}^{m} R_i \tag{3}$$

The usual three-sigma control limits for the $\bar{X}$ control chart are given by

$$LCL_{\bar{X}} = \bar{\bar{X}} - \frac{3}{d_2\sqrt{n}}\bar{R} \tag{4}$$

$$CL_{\bar{X}} = \bar{\bar{X}} \tag{5}$$

$$UCL_{\bar{X}} = \bar{\bar{X}} + \frac{3}{d_2\sqrt{n}}\bar{R} \tag{6}$$

where CL is a center line and UCL and LCL are referred to as the upper control limit and the lower control limit, respectively.

The estimators of the R chart with the usual three-sigma control limits are

$$LCL_{\bar{X}} = D_3\bar{R} \tag{7}$$

$$CL_{\bar{X}} = \bar{R} \tag{8}$$

$$UCL_R = D_4\overline{R} \tag{9}$$

The values of D_3 and D_4 for various sample sizes $2 \le n \le 25$ are also available in the textbooks or literature of quality control [19].

3 Interval numbers and united extension

In the following, we use lower case and boldface letters to denote real numbers and interval numbers or closed intervals on $\mathbb{R}$, respectively. Let $\mathbf{A} = \{a : a^l \le a \le a^u, a \in \mathbb{R}\} = [a^l, a^u]$ be a closed interval number, where a^l and a^u are the left and right endpoints of the interval $\mathbf{A}$ on the real line $\mathbb{R}$. If $a^l = a^u$, then the interval number $\mathbf{A}$ degenerates to be a real number, $\mathbf{A} = [a,a] = a$. We define the following quantities:

$$c(\mathbf{A}) = \frac{1}{2}\left(a^u + a^l\right) \text{ and } w(\mathbf{A}) = \frac{1}{2}\left(a^u - a^l\right) \tag{10}$$

where $c(\mathbf{A})$ is the center of $\mathbf{A}$ and $w(\mathbf{A})$ is the width of $\mathbf{A}$. Therefore, the interval $\mathbf{A}$ can be equivalently expressed as $\mathbf{A} = \langle c(\mathbf{A}), w(\mathbf{A})\rangle$. In order to construct the $\overline{\mathbf{X}}$ and $\mathbf{R}$ control charts for interval-valued data, the following proposition developed in the interval analysis is very useful.

Proposition 3.1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous real-valued (degenerate intervals) or even a continuous interval-valued function and let $A_1, A_2, \ldots, A_n$ be n interval numbers of $\mathbb{R}$. By the united extension defined in Moore [22-24], we can induce an interval-valued function $\tilde{f} : \mathbf{I}(\mathbb{R}) \times \mathbf{I}(\mathbb{R}) \times \ldots \mathbf{I}(\mathbb{R}) \to \mathbf{I}(\mathbb{R})$ via the real-valued function $f(a_1, a_2, \ldots, a_n)$; that is, $\tilde{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$ is a interval subset of $\mathbb{R}$. The united extension $\tilde{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$ is given by

$$\tilde{f}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n) = \bigcup_{\{(a_1,\ldots a_n):a_1^l \le a_1 \le a_1^u, \ldots, a_n^l \le a_n \le a_n^u\}} f(a_1, \ldots, a_n).$$

4 Control charts for interval-valued data

Now, we are in a position to present the $\overline{\mathbf{X}}$ and $\mathbf{R}$ control charts for monitoring the process average and variability with interval-valued sample data.

4.1 $\overline{\mathrm{X}}$ control chart

Let $\mathbf{x}_{ij}$ be interval observations (interval-valued data) for $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots, n$, which consist of m subgroups and each subgroup contains n observations. Of course, the interval-valued data are assumed as interval numbers, i.e., $\mathbf{x}_{ij} = [x_{ij}^l, x_{ij}^u]$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. With these interval-valued data, it suffices to present the constructing procedure for the interval upper and lower control limits by using the united extension principle.

From Eq. (4), $UCL_{\overline{X}}$ is a function of real sample data X_{ij}, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Therefore, we can write $UCL_{\overline{X}} = f_1(X_{ij})$, $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots, n$. In other words, for any given interval observations, the corresponding real-valued data x_{ij} for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$ can form the estimate of upper control limit $ucl_{\overline{X}} = f_1(x_{ij})$, $i = 1,2,\ldots m$ and $j = 1,2,\ldots,n$.

Based on the united extension principle, the estimate of interval upper control limit for the $\overline{\mathbf{X}}$ control chart can be obtained. For notational convenience, we denote the estimate of interval upper control limit by $\mathbf{ucl}_{\overline{\mathbf{X}}}$. By Proposition 3.1, $\mathbf{ucl}_{\overline{\mathbf{X}}}$ is an interval number, i.e, $\mathbf{ucl}_{\overline{\mathbf{X}}} = [ucl^l_{\overline{X}}, ucl^u_{\overline{X}}]$. Therefore, from Eq. (6) and Proposition 3.1, the left-endpoint $ucl^l_{\overline{X}}$ and right-end point $ucl^u_{\overline{X}}$ can be given by

$$ucl^l_{\overline{X}} = \min_{\{(x_{ij}):x^l_{ij}\le x_{ij}\le x^u_{ij}, i=1,2,\ldots,m;\, j=1,2\ldots,n\}} f_1(x_{ij}) \tag{11}$$

and

$$ucl^u_{\overline{X}} = \max_{\{(x_{ij}):x^l_{ij}\le x_{ij}\le x^u_{ij}, i=1,2,\ldots,m;\, j=1,2\ldots,n\}} f_1(x_{ij}) \tag{12}$$

Similarly, according to Eq. (4) and Proposition 3.1, the estimate of lower control limit is $lcl_{\overline{X}} = f_2(x_{ij})$, $i = 1,2,\ldots m$ and $j = 1,2,\ldots,n$. The estimate of interval lower control limit $\mathbf{lcl}_{\overline{\mathbf{X}}} = [lcl^l_{\overline{X}}, lcl^u_{\overline{X}}]$ is shown below

$$lcl^l_{\overline{X}} = \min_{\{(x_{ij}):x^l_{ij}\le x_{ij}\le x^u_{ij}, i=1,2,\ldots,m;\, j=1,2\ldots,n\}} f_2(x_{ij}) \tag{13}$$

and

$$lcl^u_{\overline{X}} = \max_{\{(x_{ij}):x^l_{ij}\le x_{ij}\le x^u_{ij}, i=1,2,\ldots,m;\, j=1,2\ldots,n\}} f_2(x_{ij}) \tag{14}$$

In order to realize whether the interval average of i-th sample lies within the interval upper and lower control limits. From Eq. (1) and Proposition 3.1, the estimate of the interval average of i-th sample is $\overline{x}_i = f_3(x_{i1}, x_{i2}, \ldots, x_{in})$, $i = 1,2,\ldots,m$, and the estimate of the interval average of i-th sample $\overline{\mathbf{x}}_i = [\overline{x}^l_i, \overline{x}^u_i]$ can be obtained by solving the formulae

$$\overline{x}^l_i = \min_{\{(x_{i1},\ldots,x_{in}):x^l_{i1}\le x_{i1}\le x^u_{i1},\ldots,x^l_{in}\le x_{i1}\le x^u_{in}, i=1,2,\ldots,m\}} f_3(x_{i1},\ldots,x_{in}) \tag{15}$$

and

$$\overline{x}^u_i = \max_{\{(x_{i1},\ldots,x_{in}):x^l_{i1}\le x_{i1}\le x^u_{i1},\ldots,x^l_{in}\le x_{i1}\le x^u_{in}, i=1,2,\ldots,m\}} f_3(x_{i1},\ldots,x_{in}) \tag{16}$$

Clearly, Eqs. (11)-(16) are nonlinear programming problems with bounded variables, the built-in optimization subroutine called ``fmincon" provided in the commercial software Matlab is used to construct the interval values in Eqs. (11)-(16).

It can be noted that the function $\overline{x}_i = f_3(x_{i1}, x_{i2}, \ldots, x_{in})$ in Eqs (15) and (16) is monotonically increasing, in this case, the expressions and computations can be simplified.

$$\overline{x}_i^l = \frac{1}{n}\sum_{j=1}^{n} x_{ij}^l \text{ and } \overline{x}_i^u = \frac{1}{n}\sum_{j=1}^{n} x_{ij}^u \text{ for } i = 1,2,\ldots m. \tag{17}$$

Note that the interval upper and lower control limits in Eqs (11)-(14) cannot possess this kind of simple expressions.

4.2 $\mathbf{R}$ control chart

From Eqs. (7) and (9) and Proposition 3.1, similarly, the estimates of upper and lower control limits are $lcl_R = f_4(x_{ij})$ and $lcl_R = f_5(x_{ij})$, $i = 1,2,\ldots m$ and $j = 1,2,\ldots,n$, respectively, and the estimates of the interval control limits for $\mathbf{R}$ control chart, i.e., $\mathbf{ucl_R} = [ucl_R^l, ucl_R^u]$ and $\mathbf{lcl_R} = [lcl_R^l, lcl_R^u]$ are

$$ucl_R^l = \min_{\{(x_{ij}):x_{ij}^l \le x_{ij} \le x_{ij}^u, i=1,2,\ldots,m; j=1,2\ldots,n\}} f_4(x_{ij}) \tag{18}$$

$$ucl_R^u = \max_{\{(x_{ij}):x_{ij}^l \le x_{ij} \le x_{ij}^u, i=1,2,\ldots,m; j=1,2\ldots,n\}} f_4(x_{ij}) \tag{19}$$

$$lcl_R^l = \min_{\{(x_{ij}):x_{ij}^l \le x_{ij} \le x_{ij}^u, i=1,2,\ldots,m; j=1,2\ldots,n\}} f_5(x_{ij}) \tag{20}$$

$$lcl_R^u = \max_{\{(x_{ij}):x_{ij}^l \le x_{ij} \le x_{ij}^u, i=1,2,\ldots,m; j=1,2\ldots,n\}} f_5(x_{ij}) \tag{21}$$

In order to realize whether the interval range of i-th sample $\mathbf{r}_i$ lies within the interval upper limit $\mathbf{ucl_R}$ and lower control limit $\mathbf{lcl_R}$. The estimate of the average of i-th sample is $r_i = f_6(x_{i1}, x_{i2}, \ldots, x_{in})$, $i = 1,2,\ldots m$. According to Eq. (3) and Proposition 3.1, the estimate of the interval average of i-th sample $\mathbf{r}_i = [r_i^l, r_i^u]$ can be obtained by solving formulae

$$r_i^l = \min_{\{(x_{i1},\ldots,x_{in}):x_{i1}^l \le x_{i1} \le x_{i1}^u,\ldots,x_{in}^l \le x_{i1} \le x_{in}^u, i=1,2,\ldots,m\}} f_6(x_{i1},\ldots,x_{in}) \tag{22}$$

and

$$r_i^u = \max_{\{(x_{i1},\ldots,x_{in}):x_{i1}^l \le x_{i1} \le x_{i1}^u,\ldots,x_{in}^l \le x_{i1} \le x_{in}^u, i=1,2,\ldots,m\}} f_6(x_{i1},\ldots,x_{in}) \tag{23}$$

Similarly, Eqs. (18)-(23) are nonlinear programming problems with bounded variables, which can be solved by using the "fmincon" function in Matlab.

5 Methods for classifying manufacturing processes

When setting up the interval $\overline{\mathbf{X}}$ and $\mathbf{R}$ control charts, it is the best to begin with the interval $\mathbf{R}$ chart. Since control limits on the interval $\overline{\mathbf{X}}$ chart depend on the process variability, unless the process variability is in control, these limits will not have much meaning.

For the purpose of realizing whether the plotted interval average $\mathbf{r}_i$ lie within the interval upper control limit $\mathbf{ucl_R}$ and interval lower control limit $\mathbf{lcl_R}$, we develop a method ranking interval numbers among the set of all interval numbers.

Let $\mathbf{I}$ be the set of all closed intervals in $\mathbb{R}$. According to Eq. (10), Sengupta and Pal [21] proposed an acceptability function $\mathcal{A}:\mathbf{I}(\mathbb{R})\times\mathbf{I}(\mathbb{R})\to[0,\infty)$

$$\mathcal{A}(\mathbf{A}\prec\mathbf{B})=\frac{c(\mathbf{B})-c(\mathbf{A})}{w(\mathbf{B})+w(\mathbf{A})} \text{ for } c(\mathbf{A})\le c(\mathbf{B}), \tag{24}$$

where $w(\mathbf{A})+w(\mathbf{B})\neq 0$. The values of $\mathcal{A}(\mathbf{A}\prec\mathbf{B})$ are interpreted as the grade of acceptability of $\mathbf{A}$ to be inferior to $\mathbf{B}$. Their possible values and interpretations are

$$\mathcal{A}(\mathbf{A}\prec\mathbf{B})=\begin{cases}=0, & if \quad c(\mathbf{A})=c(\mathbf{B})\\ \in(0,1) & if \quad c(\mathbf{A})=c(\mathbf{B}) \quad and \quad a^u>b^l\\ \ge 1 & if \quad c(\mathbf{A})=c(\mathbf{B}) \quad and \quad a^u\le b^l\end{cases} \tag{25}$$

- If $\mathcal{A}(\mathbf{A}\prec\mathbf{B})=0$, then the premise "$\mathbf{A}$ is inferior to $\mathbf{B}$" is not accepted.
- If $\mathcal{A}(\mathbf{A}\prec\mathbf{B})\in(0,1)$, then the decision-maker accepts the premise $\mathbf{A}\prec\mathbf{B}$ with a satisfaction grade ranging from 0 to 1.
- If $\mathcal{A}(\mathbf{A}\prec\mathbf{B})\ge 1$, then the decision-maker is absolutely satisfied with the premise $\mathbf{A}\prec\mathbf{B}$. If the value of $\mathcal{A}(\mathbf{A}\prec\mathbf{B})$ is large, then $\mathbf{B}$ is superior to $\mathbf{A}$ in a very strong sense.

5.1 $\mathbf{R}$ control chart

Considered the process variability, if the value of $\mathcal{A}(\mathbf{ucl_R}\prec\mathbf{r}_i)$ is large, e.g., $\mathcal{A}(\mathbf{ucl_R}\prec\mathbf{r}_i)\ge 1$, then it implies that the process is out of control. Conversely, if the value of $\mathcal{A}(\mathbf{r}_i\prec\mathbf{ucl_R})$ is large, then, theoretically, the manufacturing process seems good, since the process variability is small in the interval sense. However, under this condition, the decision-maker should be cautious, because the result often does not represent a real improvement in the manufacturing process. In many situations, it may be caused by errors in the inspection process due to inadequately trained or inexperienced inspectors or the inspectors deliberately ignoring the nonconforming items or reporting the fictitious data. Therefore, the decision-maker still has to check carefully what the assignable causes are. Sometimes, if the manufacturing process really improves, the decision-maker can take this chance to record all the settings of current manufacturing process for the future reference. Finally, two other rules are suggested as follows:

- if $A(\mathbf{r}_i\prec\mathbf{lcl_R})=\gamma\in(0,1)$ and the set-up cost is small, then the manufacturing process is intervened to record the beneficial factors or to remove the non-beneficial causes;
- if $A(\mathbf{r}_i\prec\mathbf{lcl_R})=\gamma\in(0,1)$ and the set-up cost outweighs the expected quality gains, the manufacturing process continues.

Under the above rules, the conditions of the manufacturing process are categorized as follows.

(i) The process is in-control if $\mathcal{A}(\mathbf{r}_i\prec\mathbf{ucl_R})\ge 1$ and $\mathcal{A}(\mathbf{lcl_R}\prec\mathbf{r}_i)\ge 1$.

(ii) The process is out-of-control if $\mathcal{A}(\mathbf{ucl_R}\prec\mathbf{r}_i)\ge 1$ or $\mathcal{A}(\mathbf{r}_i\prec\mathbf{lcl_R})\ge 1$.

(iii) The process is rather out-of-control with degree $\gamma\in[0,1)$ if $\mathcal{A}(\mathbf{ucl_R}\prec\mathbf{r}_i)=\gamma$ or $\mathcal{A}(\mathbf{r}_i\prec\mathbf{lcl_R})=\gamma$.

(iv) The process is rather in-control with degree $\gamma\in[0,1)$ if one of the following situations is observed:

- $\mathcal{A}(\mathbf{r}_i\prec\mathbf{ucl_R})\ge 1$ and $\mathcal{A}(\mathbf{lcl_R}\prec\mathbf{r}_i)=\gamma$.

- $\mathrm{A}\,(\mathbf{lcl}_{\mathbf{R}} \prec \mathbf{r}_i) \ge 1$ and $\mathrm{A}\,(\mathbf{r}_i \prec \mathbf{ucl}_{\mathbf{R}}) = \gamma$.
- $\mathrm{A}\,(\mathbf{r}_i \prec \mathbf{ucl}_{\mathbf{R}}) = \gamma_1$ and $\mathrm{A}\,(\mathbf{lcl}_{\mathbf{R}} \prec \mathbf{r}_i) = \gamma_2$, where $\gamma_1, \gamma_2 \in (0,1)$ and $\gamma = \min\{\gamma_1, \gamma_2\}$.

5.2 $\overline{\mathrm{X}}$ control chart

Similarly, we can categorize the manufacturing process by monitoring $\overline{\mathbf{X}}$ control chart as follows:

(i) The process is in-control if $\mathrm{A}\,(\overline{\mathbf{x}}_i \prec \mathbf{ucl}_{\overline{\mathbf{X}}}) \ge 1$ and $\mathrm{A}\,(\mathbf{lcl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i) \ge 1$.

(ii) The process is out-of-control if $\mathrm{A}\,(\mathbf{ucl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i) \ge 1$ or $\mathrm{A}\,(\overline{\mathbf{x}}_i \prec \mathbf{lcl}_{\overline{\mathbf{X}}}) \ge 1$.

(iii) The process is rather out-of-control with degree $\gamma \in [0,1)$ if $\mathrm{A}\,(\mathbf{ucl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i) = \gamma$ or $\mathrm{A}\,(\overline{\mathbf{x}}_i \prec \mathbf{lcl}_{\overline{\mathbf{X}}}) = \gamma \in (0,1)$.

(iv) The process is rather in-control with degree $\gamma \in [0,1)$ if one of the following situations is observed:

- $\mathrm{A}\,(\overline{\mathbf{x}}_i \prec \mathbf{lcl}_{\overline{\mathbf{X}}}) \ge 1$ and $\mathrm{A}\,(\mathbf{lcl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i) = \gamma$.
- $\mathrm{A}\,(\mathbf{lcl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i) \ge 1$ and $\mathrm{A}\,(\overline{\mathbf{x}}_i \prec \mathbf{ucl}_{\overline{\mathbf{X}}}) = \gamma$.
- $\mathrm{A}\,(\overline{\mathbf{x}}_i \prec \mathbf{ucl}_{\overline{\mathbf{X}}}) = \gamma_1$ and $\mathrm{A}\,(\mathbf{lcl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i) = \gamma_2$, where $\gamma_1, \gamma_2 \in (0,1)$ and $\gamma = \min\{\gamma_1, \gamma_2\}$.

In the sequel, the $\overline{\mathbf{X}}$ and $\mathbf{R}$ control charts with practical interval-valued data are provided to illustrate the applicability of proposed methodologies.

6 A practical example

The applications of Light Emitting Diodes (LEDs) are growing rapidly, since the long life span and high intensity of solid-state illumination with wide range of colors have been recently developed and become available, which enabled the applications of LEDs in a wide variety of areas such as automotive lighting, computer displays, liquid crystal display televisions, signaling and general lighting, etc. The example investigated here is taken from a LED-based lighting fixture (LED-LF) manufacturer, which is located on Tainan Industrial Park in Taiwan.

The luminous intensity of LED sources is the critical characteristic for one type of LED-LFs, which highly determines their conforming level of the LED-LFs. The luminous intensity value is generally rated in terms of millicandela (mcd). All light measurements and rating systems until now somewhat depend on the perception of the human eye, or imprecise terminology and calibration standards [25]. During the period of inspection, limitations of human's sensitivity, visual perception, visual fatigues, and inconsistent detection cause the indispensable subjectivity. That is, the randomness is not the only aspect of uncertainty for the fraction of nonconforming data of the LED-LFs. The occurrence of vagueness provides another uncertainty that should be taken into account in the problem.

The LED fabrication process is used in conjunction with the luminous intensity of one type of LED-LFs manufacturing. The time interval between samples or subgroups is one hour. The interval-valued data (interval numbers) $\mathbf{x}_{ij}$ for $i = 1, 2, \ldots, 24$ and $j = 1, 2, \ldots, 4$ (twenty-four samples (subgroups), where each sample (subgroup) has size four) have been collected when we think the process is in control. These interval-valued data of the luminous intensity values of LED-LFs are shown in Table 1. We wish to establish the statistical control of the luminous intensity by using the interval $\overline{\mathbf{X}}$ and $\mathbf{R}$ control charts.

To find the control limits on the interval $\mathbf{R}$ chart, we use D_4 =2.114 and D_3 =0 for samples of size n =4 (see Montgomery [19]). With the data in Table 1 and executing Eqs. (18)-(24), we have the interval upper control limit $\mathbf{ucl}_{\mathbf{R}}$, the interval lower control limit $\mathbf{lcl_R}$ and the interval range of each interval sample data $\mathbf{r}_i$ for $i = 1, 2, \ldots, 24$ which are shown in Fig. 1. The interval values of $\mathbf{ucl_R}$, $\mathbf{lcl_R}$ and $\mathbf{r}_i$ and the values of the acceptability function $\mathbb{A}(\mathbf{r}_i \prec \mathbf{ucl_R})$, $\mathbb{A}(\mathbf{ucl_R} \prec \mathbf{r}_i)$, $\mathbb{A}(\mathbf{r}_i \prec \mathbf{lcl_R})$, and $\mathbb{A}(\mathbf{lcl_R} \prec \mathbf{r}_i)$ for $i = 1, 2, \ldots, 24$ are also listed in Table 2, indicating the process variability being in control.

Table 1. The luminous intensity values ($\times 10^3$ mcd) in the LED fabrication process

Sample Number i	**LED-LFs**			
	1	**2**	**3**	**4**
1	[22.51,24.60]	[22.57,23.22]	[27.12,28.52]	[25.26,27.22]
2	[27.75,30.29]	[25.22,28.03]	[27.20,32.11]	[26.02,28.24]
3	[27.14,31.51]	[21.53,23.04]	[24.79,24.83]	[24.22,26.03]
4	[27.40,28.10]	[24.19,24.24]	[23.67,25.03]	[24.63,25.93]
5	[25.25,27.22]	[28.54,29.93]	[22.26,25.49]	[22.65,25.15]
6	[23.22,24.01]	[25.14,25.68]	[25.67,26.36]	[23.06,25.16]
7	[24.07,24.48]	[23.47,29.02]	[22.02,25.42]	[24.66,28.06]
8	[25.29,27.95]	[26.25,28.12]	[22.07,25.95]	[24.97,25.25]
9	[26.58,28.58]	[20.62,22.02]	[25.11,25.54]	[27.33,27.45]
10	[24.04,24.52]	[24.76,27.60]	[24.42,26.25]	[25.35,26.21]
11	[23.36,24.55]	[23.55,24.31]	[25.70,25.77]	[22.15,26.46]
12	[27.43,28.19]	[25.19,25.24]	[23.67,25.03]	[24.63,25.93]
13	[24.32,27.47]	[25.97,26.22]	[22.60,24.17]	[27.16,27.49]
14	[22.61,24.70]	[22.67,23.32]	[27.02,28.42]	[25.26,27.22]
15	[26.30,27.39]	[25.63,27.08]	[24.67,25.47]	[26.74,27.54]
16	[26.26,29.94]	[26.31,26.50]	[26.42,29.66]	[21.93,24.01]
17	[23.96,25.17]	[28.23,28.35]	[26.09,28.92]	[24.23,26.15]
18	[25.59,29.00]	[24.92,27.11]	[24.27,28.36]	[24.32,25.24]
19	[24.21,28.94]	[25.77,26.23]	[24.61,24.88]	[26.06,28.33]
20	[26.92,29.02]	[23.21,25.19]	[23.03,26.45]	[26.28,27.29]
21	[25.24,25.75]	[27.07,31.26]	[27.11,29.12]	[28.05,28.13]
22	[28.79,31.25]	[27.25,28.44]	[27.04,27.73]	[28.21,28.96]
23	[23.22,25.19]	[25.76,27.10]	[24.19,26.00]	[25.30,27.96]
24	[24.11,26.62]	[23.00,25.22]	[21.27,22.37]	[17.26,18.62]

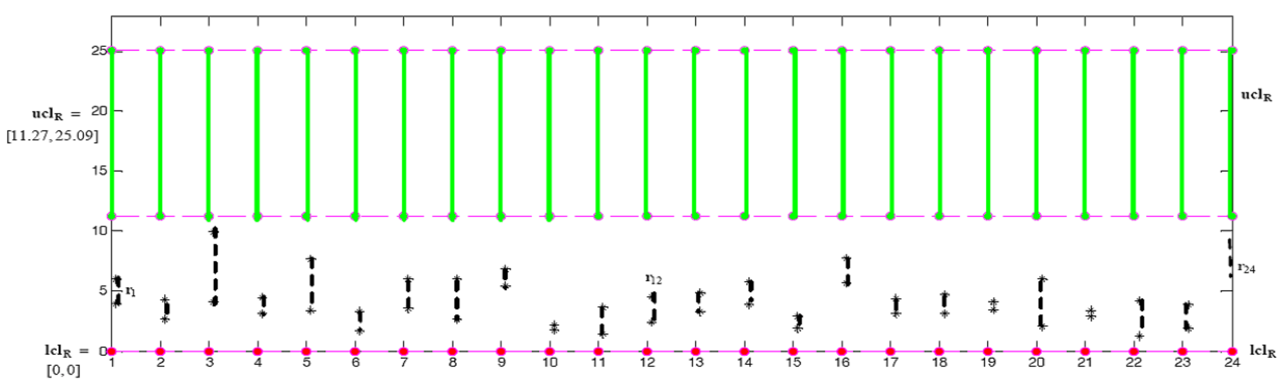

Figure 1. The $\mathbf{R}$ control chart for the LEDs fabrication process.

Therefore, we may now construct the $\overline{\mathbf{X}}$ control chart. To determine the interval control limits for the $\overline{\mathbf{X}}$ chart, we use d_2 =2.059 for samples of size n =4 (see Montgomery [19]). Based on Eqs. (11)-(16) and (24), the interval upper control limit $\mathbf{ucl}_{\overline{\mathbf{X}}}$, the interval lower control limit $\mathbf{lcl}_{\overline{\mathbf{X}}}$ and the interval average of each interval sample data $\overline{\mathbf{x}}_i$ for $i = 1,2,\ldots,24$ are obtained and shown in Fig. 2. The interval values of $\mathbf{ucl}_{\overline{\mathbf{X}}}$, $\mathbf{lcl}_{\overline{\mathbf{X}}}$ and $\overline{\mathbf{x}}_i$ and the values of the acceptability function $\mathcal{A}(\overline{\mathbf{x}}_i \prec \mathbf{ucl}_{\overline{\mathbf{X}}})$, $\mathcal{A}(\mathbf{ucl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i)$, $\mathcal{A}(\overline{\mathbf{x}}_i \prec \mathbf{lcl}_{\overline{\mathbf{X}}})$, and $\mathcal{A}(\mathbf{lcl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i)$ for $i = 1,2,\ldots,24$ are also listed in Table 3.

Table 2. $\mathbf{R}$ control chart

Sample Number i	$\mathbf{ucl}_{\mathbf{R}} = [11.27, 25.09]$ $\mathbf{lcl}_{\mathbf{R}} = [0,0]$					
	$\mathbf{r}_i$	$\mathcal{A}(\mathbf{r}_i \prec \mathbf{ucl}_{\mathbf{R}})$	$\mathcal{A}(\mathbf{ucl}_{\mathbf{R}} \prec \mathbf{r}_i)$	$\mathcal{A}(\mathbf{r}_i \prec \mathbf{lcl}_{\mathbf{R}})$	$\mathcal{A}(\mathbf{lcl}_{\mathbf{R}} \prec \mathbf{r}_i)$	**Process Status**
1	[4.00,6.01]	1.66	--	--	4.98	In Control
2	[2.64,4.27]	1.91	--	--	4.24	In Control
3	[4.10,9.98]	1.13	--	--	2.39	In Control
4	[3.16,4.43]	1.91	--	--	5.98	In Control
5	[3.39,7.67]	1.40	--	--	2.58	In Control
6	[1.67,3.30]	2.03	--	--	3.05	In Control
7	[3.58,6.04]	1.64	--	--	3.91	In Control
8	[2.70,6.05]	1.61	--	--	2.61	In Control
9	[5.43,6.83]	1.58	--	--	8.76	In Control
10	[1.73,2.17]	2.28	--	--	8.86	In Control
11	[1.46,3.62]	1.96	--	--	2.35	In Control
12	[2.40,4.52]	1.85	--	--	3.26	In Control
13	[3.30,4.89]	1.83	--	--	5.15	In Control
14	[3.90,5.81]	1.69	--	--	5.08	In Control
15	[1.92,2.87]	2.14	--	--	5.04	In Control
16	[5.65,7.73]	1.45	--	--	6.43	In Control
17	[3.18,4.39]	1.92	--	--	1.92	In Control
18	[3.12,4.73]	1.85	--	--	1.85	In Control
19	[3.45,4.12]	1.99	--	--	1.99	In Control
20	[2.10,5.99]	1.60	--	--	1.60	In Control
21	[2.89,3.37]	2.18	--	--	13.04	In Control
22	[1.23,4.21]	1.84	--	--	1.83	In Control
23	[1.91,3.88]	1.94	--	--	2.94	In Control
24	[6.60,9.36]	1.23	--	--	5.78	In Control

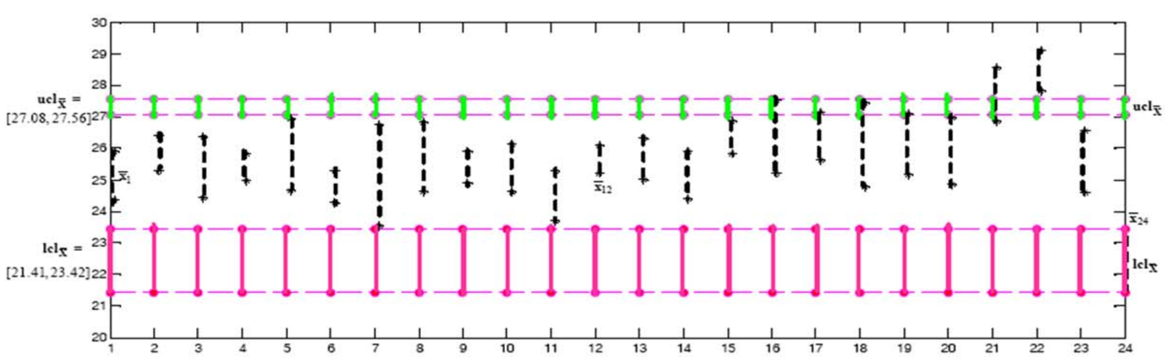

Figure 2. The $\overline{\mathbf{X}}$ control chart for the LEDs fabrication process.

Table 3. $\overline{\mathbf{X}}$ control chart

Sample Number i	$\mathbf{ucl}_{\overline{\mathbf{X}}} = [27.08, 27.56]$ $\mathbf{lcl}_{\overline{\mathbf{X}}} = [21.41, 23.42]$					
	$\overline{\mathbf{x}}_i$	$\mathbb{A}(\overline{\mathbf{x}}_i \prec \mathbf{ucl}_{\overline{\mathbf{X}}})$	$\mathbb{A}(\mathbf{ucl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i)$	$\mathbb{A}(\overline{\mathbf{x}}_i \prec \mathbf{lcl}_{\overline{\mathbf{X}}})$	$\mathbb{A}(\mathbf{lcl}_{\overline{\mathbf{X}}} \prec \overline{\mathbf{x}}_i)$	**Process Status**
1	[24.37,25.89]	2.18	--	--	1.53	In Control
2	[25.30,26.42]	1.82	--	--	2.19	In Control
3	[24.42,26.35]	1.60	--	--	1.50	In Control
4	[24.97,25.83]	2.87	--	--	2.08	In Control
5	[24.68,26.95]	1.09	--	--	1.58	In Control
6	[24.27,25.30]	3.34	--	--	1.56	In Control
7	[23.56,26.75]	1.18	--	--	1.05	In Control
8	[24.65,26.82]	1.19	--	--	1.58	In Control
9	[24.91,25.90]	2.60	--	--	1.99	In Control
10	[24.64,26.15]	1.93	--	--	1.69	In Control
11	[23.69,25.27]	2.74	--	--	1.15	In Control
12	[25.23,26.10]	2.44	--	--	2.25	In Control
13	[25.01,26.34]	1.81	--	--	1.95	In Control
14	[24.39,25.92]	2.15	--	--	1.54	In Control
15	[25.84,26.87]	1.27	--	--	2.58	In Control
16	[25.23,27.53]	0.68	--	--	1.84	Rather in Control with Degree γ =0.68
17	[25.63,27.15]	0.93	--	--	2.24	Rather in Control with Degree γ =0.93
18	[24.7827.43]	0.78	--	--	1.58	Rather in Control with Degree γ =0.78
19	[25.16,27.10]	0.98	--	--	1.88	Rather in Control with Degree γ =0.98
20	[24.86,26.99]	1.07	--	--	1.69	In Control
21	[26.87,28.57]	--	0.36	--	2.85	Rather out of Control with Degree γ =0.36
22	[27.82,29.10]	--	1.30	--	3.67	Out of Control
23	[24.62,26.56]	1.42	--	--	1.60	In Control
24	[21.41,23.21]	4.38	--	0.06	--	Rather out of Control with Degree γ =0.06

For subgroup 16, 17, 18 and 19, the manufacturing process is categorized as rather in control with degree γ =0.68, γ =0.93, γ =0.78 and γ =0.98. For subgroup 21, the manufacturing process is categorized as rather out of control with degree γ =0.36. In these cases, if the setup cost is small (or tolerable for the decision-maker), intervening the manufacturing process is suggested. For subgroup 22, the process is categorized as out of control; that is, special causes have occurred in the underlying manufacturing process.

7 Conclusions

An interval number can be thought as an extension of the concept of a real number, which signifies the extent of tolerance that the quality characteristic can possibly take. For monitoring and controlling the interval-valued data in the manufacturing or service processes, a certain degree of uncertainty other than randomness should also be incorporated in the analysis in order to avoid potential bias and loss of efficiency. In this paper, an interval-data analysis methodology is employed to construct $\overline{\mathbf{X}}$ and $\mathbf{R}$ control charts whose interval control limits are obtained by using the united extension principle inherited from the interval number analysis. Then, for identifying the special causes of variation and alarming the requirement for corrective actions existing in the underlying processes, an acceptability function is employed to compare and the set of closed interval numbers. The acceptability function approach provides the ability of making linguistic decisions like ``rather in control'' or ``rather out of control''. If the process is classified as ``rather in control'' and the setup cost is small or tolerable, intervening the manufacturing process to make the products more reliable is suggested. On the other hand, if the process is classified as "rather out of control" and the setup cost is large, it may not be required to interrupt the manufacturing process unless expected quality gains of products outweigh the cost. This kind of intermediate classifications can supplement the shortcomings of binary classifications of conventional Shewhart control charts when interval-valued data inevitably appear in the manufacturing or service processes.

References

[1] N. Sugano, "Fuzzy set theoretical approach to achromatic relevant color on the natural color system," International Journal of Innovative Computing, Information and Control. 2006; 2(1): 193-203.

[2] K. Wang and F. Tsung, "Run-to-run process adjustment using categorical observations," Journal of Quality Technology. 2007; 39(4): 312-325.

[3] M. B. Miles and A. M. Huberman, Qualitative Data Analysis: an Expanded Sourcebook. 2nd, Thousand Oaks, California, 1994.

[4] J. H. Wang, and T. Raz, "On the construction of control charts using linguistic variables," International Journal of Production Research. 1990; 28(3): 477-487. http://dx.doi.org/10.1080/00207549008942731

[5] Kanagawa, F. Tamaki, and H. Ohta, "Control charts for process average and variability based on linguistic data," International Journal of Production Research. 1993; 31(4): 913-922. http://dx.doi.org/10.1080/00207549308956765

[6] Eichhorn, D. Girimonte, A. Klose, and R. Kruse, "Soft computing for automated surface quality analysis of exterior car body panels," Applied Soft Computing. 2005; 5(3): 301-313. http://dx.doi.org/10.1016/j.asoc.2004.08.002

[7] R. Seising, "On the absence of strict boundaries-vagueness, haziness, and fuzziness in philosophy, science, and medicine," Applied Soft Computing. 2008; 8(3): 1232-1242. http://dx.doi.org/10.1016/j.asoc.2007.02.017

[8] R. S. Berns, Billmeyer and Saltzman's Principles of Color Technology. 3rd, Wiley-Interscience, 2000.

[9] R. Filzmoser and R. Vertl, "Testing hypotheses with fuzzy data: the fuzzy p-value," Metrika. 2004; 59: 21-29. http://dx.doi.org/10.1007/s001840300269

[10] R. Viertl and D. Hareter, "Fuzzy estimation and imprecise probability," Journal of Applied Mathematics and Mechanics. 2004; 84 (10-11): 731-739.

[11] P. Giordani and H. A. L. Kiers, "A comparison of three methods for principal component analysis of fuzzy interval data," Computational Statistics & Data Analysis. 2006; 51: 379-397. http://dx.doi.org/10.1016/j.csda.2006.02.019

[12] C.J. Spanos and R. L. Chen, "Using qualitative observations for process tuning and control," IEEE Transactions on Semiconductor Manufacturing. 1997; 10(2): 307-316. http://dx.doi.org/10.1109/66.572086

[13] P. D. Fasulo, W. R. Rodgers, R. A. Ottaviani, and D. L. Hunter, "Extrusion processing of TPO nanocomposites," Polymer Engineering and Science. 2004; 44(6): 1036-1045. http://dx.doi.org/10.1002/pen.20097

[14] W. M. K. Trochim and J. P. Donnelly, The Research Methods Knowledge Base. 3th, Automatic Dog Publishing, 2006.

[15] L. Jaulin, M. Kieffer, O. Didri, and E. Walter, Applied Interval Analysis. Springer, 2001. http://dx.doi.org/10.1007/978-1-4471-0249-6

[16] W. A. Lodwick and K. D. Jamison, "Special issue: interfaces between fuzzy set theory and interval analysis," Fuzzy Sets and Systems. 2003; 135(1): 1-212 (1-3).

[17] Neumaier, Interval Methods for Systems of Equations. Cambridge University Press, 1990.

[18] J. R. Evans, and W. M. Lindsay, The Management and Control of Quality. South-Western College Publishing, 2005.

[19] D. C. Montgomery, Introduction to Statistical Quality Control. 5th Ed., John Wiley \& Sons, Inc., 2005.

[20] W. H. Woodall, and D.C. Montgomery, "Research issues and ideas in statistical process control," Journal of Quality Technology. 1999; 31(4): 376-385.

[21] Sengupta and T. K. Pal, "On comparing interval numbers," European Journal of Operational Research. 2000; 127: 28-43. http://dx.doi.org/10.1016/S0377-2217(99)00319-7

[22] R. E. Moore, Interval Analysis and Automatic Error Analysis in Digital Computing. Ph.D. Thesis, Stanford University, 1962.

[23] R. E. Moore, Interval Analysis. Prentice-Hall, Inc., 1966.

[24] R. E. Moore, Methods and Applications of Interval Analysis, Society for Industrial and Applied Statistics. Philadelphia, 1979.

[25] Ryer, The light measurement handbook. International Light, Newburyport, MA, 1997.