ORIGINAL RESEARCH

Multisequent Gentzen Deduction Systems for B_2^2-valued first-order logic

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ABSTRACT

For the four-element Boolean algebra $\mathbf{B}_2^2$, a multisequent $\Gamma|\Delta|\Sigma|\Pi$ is a generalization of sequent $\Gamma \Rightarrow \Delta$ in traditional $\mathbf{B}_2$-valued first-order logic. By defining the truth-values of quantified formulas, a Gentzen deduction system $\mathbf{G}_2^2$ for $\mathbf{B}_2^2$-valued first-order logic will be built and its soundness and completeness theorems will be proved.

Key Words: Multisequent, Hypersequent, $\mathbf{B}_2^2$-valued semantics, Soundness theorem, Completeness theorem

1. INTRODUCTION

In traditional propositional logic,[1] the negation connective $\neg$ is eliminated via the following rules in a Gentzen deduction system:

$$(\neg^L)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\neg^R)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}.$$

The truth-value of a sequent $\Gamma \Rightarrow \Delta$ under an assignment v is defined by a condition of $(\exists \vee \exists)$-form, that is, either some formula A in Γ is false, or some formula B in Δ is true.

In $\mathbf{B}_2^2$-valued first-order logic, where $\mathbf{B}_2^2 = (\{\mathtt{t}, \top, \bot, \mathtt{f}\}, \cap, \cup)$ is the four-element Boolean algebra, a multisequent is a quandrupe $(\Gamma, \Delta, \Sigma, \Pi)$, where $\Gamma, \Delta, \Sigma, \Pi$ are sets of formulas. Multisequent $(\Gamma, \Delta, \Sigma, \Pi)$ is true in a model M and an assignment v if either

- some formula A in Γ has truth-value $\mathtt{t}$, or
- some formula B in Δ has truth-value $\top$, or
- some formula C in Σ has truth-value $\bot$, or
- some formula D in Π has truth-value $\mathtt{f}$.

By the semantics, multisequents are different from hypersequents.[2,3] A sequent $\Gamma \Rightarrow \Delta$ is taken as a multisequent $\Delta|\Gamma$.

Here, negation $\neg$ commutes $\mathtt{t}$ with $\mathtt{f}$ and $\top$ with $\bot$. Traditional deduction rules $(\neg^L)$ and $(\neg^R)$ do not work here.

For quantifiers, in traditional first-order logic, $\forall x A(x)$ is true if for each element a, $A(x/a)$ is true; and $\forall x A(x)$ is false if for some element a, $A(x/a)$ is false. In $\mathbf{B}_2^2$-valued first-order logic, we define that $\forall x A(x)$ has truth-value

- $\mathtt{t}$ if for each element a, $A(x/a)$ has truth-value $\mathtt{t}$;
- $\top$ if for some element b, $A(x/b)$ has truth-value $\top$, and for each element a, $A(x/a)$ has truth-value either $\mathtt{t}$ or $\top$;
- $\bot$ if for some element b, $A(x/b)$ has truth-value $\bot$, and for

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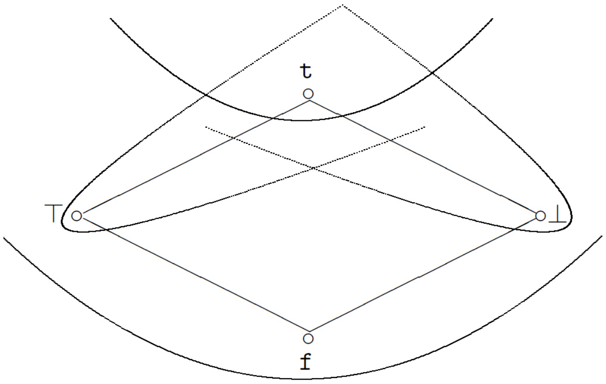
each element $a, A(x/a)$ has truth-value either $\mathtt{t}$ or $\perp$;

• $\mathtt{f}$ if for some element $a, A(x/a)$ has truth-value $\mathtt{f}$.



In this paper, we will give a sound and complete Gentzen deduction system $\mathbf{G}_2^2$[4–9] for $\mathbf{B}_2^2$-valued first-order logic and prove soundness and completeness theorems for $\mathbf{B}_2^2$-valued first-order logic, that is, for any multisequent $\Gamma|\Delta|\Sigma|\Pi$,

• Soundness theorem: if $\Gamma|\Delta|\Sigma|\Pi$ is provable in $\mathbf{G}_2^2$ then $\Gamma|\Delta|\Sigma|\Pi$ is valid;

• Completeness theorem: if $\Gamma|\Delta|\Sigma|\Pi$ is valid then $\Gamma|\Delta|\Sigma|\Pi$ is provable in $\mathbf{G}_2^2$.

The paper is organized as follows: the next section gives basic definitions in $\mathbf{B}_2$-valued first-order logic; the third section gives basic definitions of $\mathbf{B}_2^2$-valued first-order logic; the fourth section gives Gentzen deduction system $\mathbf{G}_2^2$ for $\mathbf{B}_2^2$-valued first-order logic and prove soundness and completeness theorem; the fifth section discusses the different constructions of trees in the proof of completeness theorem, and the last section concluded the paper.

2. MULTISEQUENT DEDUCTION SYSTEM FOR B_2-VALUED FIRST-ORDER LOGIC

Let L be a logical language of first-order logic which contains the following symbols:

• constant symbols: $c_0, c_1, ...$;

• variable symbols: $x_0, x_1, ...$;

• function symbols: $f_0, f_1, ...$;

• predicate symbol: $p_0, p_1, ...$; and

• logical connectives and quantifiers: $\neg, \wedge, \vee, \forall$.

A term t is a string of the following forms:

$$t ::= c|x|f(t_1, ..., t_n),$$

where f is an n-ary function symbol.

A formula A is a string of the following forms:

$$A ::= p(t_1, ..., t_n)|\neg A_1|A_1 \wedge A_2|A_1 \vee A_2|\forall x A_1(x),$$

where p is an n-ary predicate symbol.

Let $\mathbf{B}_2 = (\{\mathtt{t}, \mathtt{f}\}, \neg, \cup, \cap)$ be the least Boolean algebra, where

	$\neg$	$\cap$	t	f	$\cup$	t	f
t	f	t	t	f	t	t	t
f	t	f	f	f	f	t	f

A model M is a pair (U, I), where U is a universe and I is an interpretation such that for any constant symbol $c, I(c) \in U$; for any n-ary function symbol $f, I(f) : U^n \to U$ is a function; and for any n-ary predicate symbol $p, I(p) : U^n \to \mathbf{B}_2$ is a relation on U.

An assignment v is a function from variables to U. The interpretation $t^{I,v}$ of t in (M, v) is

$$t^{I,v} = \begin{cases} I(c) & \text{if } t = c \\ v(x) & \text{if } t = x \\ I(f)(t_1^{I,v}, ..., t_n^{I,v}) & \text{if } t = f(t_1, ..., t_n). \end{cases}$$

Given a formula A, define

$$v(A) = \begin{cases} I(p)(t_1^{I,v}, ..., t_n^{I,v}) & \text{if } A = p(t_1, ..., t_2) \\ \neg(A_1^{I,v}) & \text{if } A = \neg A_1 \\ v(A_1) \cap v(A_2) & \text{if } A = A_1 \wedge A_2 \\ v(A_1) \cup v(A_2) & \text{if } A = A_1 \vee A_2 \\ \min\{v_{x/a}(A_1(x)) : a \in U\} & \text{if } A = \forall x A_1(x) \end{cases}$$

where for any variable y,

$$v_{x/a}(y) = \begin{cases} v(y) & \text{if } y \neq x \\ a & \text{otherwise.} \end{cases}$$

Hence,

$$v(\forall x A_1(x)) = \mathtt{t} \text{ iff } \mathbf{A}a \in U(v_{x/a}(A_1(x)) = \mathtt{t})$$
$$v(\forall x A_1(x)) = \mathtt{f} \text{ iff } \mathbf{E}a \in U(v_{x/a}(A_1(x)) = \mathtt{f}),$$

where in syntax, we use $\neg, \wedge, \to, \forall, \exists$ to denote logical connectives and quantifiers; and in semantics we use $\sim, \&, \Rightarrow, \mathbf{A}, \mathbf{E}$ to denote the corresponding connectives and quantifiers.

A formula A is satisfied in (I, v), denoted by $I, v \models A$, if $v(A) = \mathtt{t}$; A is valid in I, denoted by $I \models A$, if for any assignment $v, I, v \models A$; and A is valid, denoted by $\models A$, if for any interpretation $I, I \models A$.

Let Δ, Γ be sets of formulas. A multisequent δ is of form $\Gamma|\Delta$. We say that δ is satisfied in an interpretation I and an assignment v, denoted by $I, v \models \Gamma|\Delta$, if either $I, v \models \Gamma$, or $I, v \models \Delta$, where $I, v \models \Delta$ if $v(A) = \mathtt{t}$ for some $A \in \Delta$; and $I, v \models \Gamma$ if $v(B) = \mathtt{f}$ for some $B \in \Gamma$.

δ is satisfied in an interpretation I, denoted by $I \models \delta$ if $I, v \models \delta$ for any assignment v; and δ is valid, denoted by $\models \delta$, if δ is satisfied in any interpretation I.

Gentzen deduction system $\mathbf{G}_2$ contains the following axioms and deduction rules:

• **Axioms:**

$$\frac{\Gamma \cap \Delta \neq \emptyset}{\Delta|\Gamma}(\mathbf{A}),$$

where Δ, Γ are sets of atomic formulas.

• **Deduction rules**:

$$(\neg^L)\ \frac{\Delta|A,\Gamma}{\Delta,\neg A|\Gamma} \qquad (\neg^R)\ \frac{\Delta,B|\Gamma}{\Delta|\neg B,\Gamma}$$

$$(\wedge^L)\ \frac{\Delta,A_1|\Gamma\ \ \Delta,A_2|\Gamma}{\Delta,A_1\wedge A_2|\Gamma} \qquad (\wedge^R_1)\ \frac{\Delta|B_1,\Gamma}{\Delta|B_1\wedge B_2}$$

$$(\wedge^R_2)\ \frac{\Delta|B_2,\Gamma}{\Delta|B_1\wedge B_2}$$

$$(\vee^L_1)\ \frac{\Delta,A_1|\Gamma}{\Delta,A_1\vee A_2|\Gamma} \qquad (\vee^R)\ \frac{\Delta|B_1,\Gamma\ \ \Delta|B_2,\Gamma}{\Delta|B_1\vee B_2}(\vee^R)$$

$$(\vee^L_2)\ \frac{\Delta,A_2|\Gamma}{\Delta,A_1\vee A_2|\Gamma}$$

$$(\forall^L)\ \frac{\Delta,A(x)|\Gamma}{\Delta,\forall xA(x)|\Gamma} \qquad (\forall^R)\ \frac{\Delta|B(t),\Gamma}{\Delta|\forall xB(x),\Gamma}$$

where x is a new variable not occurring free in $\forall xA(x)$, and t is a term.

Theorem 2.1 (Soundness theorem). For any multisequent $\Delta|\Gamma$, if $\vdash \Delta|\Gamma$ then $\models \Delta|\Gamma$.

Theorem 2.2 (Completeness theorem). For any multisequent $\Delta|\Gamma, \vdash \Delta|\Gamma$ only if $\models \Delta|\Gamma$.

3. $\mathbf{B}_2^2$-VALUED FIRST-ORDER LOGIC

Let $\mathbf{B}_2^2$ be a Boolean algebra $(\{\mathtt{t},\top,\bot,\mathtt{f}\},\neg,\cup,\cap)$, where

	¬	∩	t	⊤	⊥	f	∪	t	⊤	⊥	f
t	f	t	t	⊤	⊥	f	t	t	t	t	t
⊤	⊥	⊤	⊤	⊤	f	f	⊤	t	⊤	t	⊤
⊥	⊤	⊥	⊥	f	⊥	f	⊥	t	t	⊥	⊥
f	t	f	f	f	f	f	f	t	⊤	⊥	f

A model M is a pair (U, I), where U is a universe and I is an interpretation such that for any constant symbol $c, I(c) \in U$; for any n-ary function symbol $f, I(f) : U^n \to U$ is a function; and for any n-ary predicate symbol $p, I(p) : U^n \to \mathbf{B}_2^2$ is a relation on U.

Given a formula A, define

$$v(A) = \begin{cases} I(p)(t_1^{I,v},...,t_n^{I,v}) & \text{if } A = p(t_1,...,t_2) \\ \neg(A_1^{I,v}) & \text{if } A = \neg A_1 \\ v(A_1)\cap v(A_2) & \text{if } A = A_1 \wedge A_2 \\ v(A_1)\cup v(A_2) & \text{if } A = A_1 \vee A_2 \\ \text{defined below} & \text{if } A = \forall xA_1(x) \end{cases}$$

where

$$v(\forall xA_1(x)) = \begin{cases} \mathtt{t} & \text{if } \mathbf{A}a \in U(v_{x/a}(A_1(x)) = \mathtt{t}) \\ \top & \text{if } \mathbf{A}a \in U(v_{x/a}(A_1(x)) \in \{\mathtt{t},\top\})\&\mathbf{E}a \in U(v_{x/a}(A_1(x)) = \top) \\ \bot & \text{if } \mathbf{A}a \in U(v_{x/a}(A_1(x)) \in \{\mathtt{t},\bot\})\&\mathbf{E}a \in U(v_{x/a}(A_1(x)) = \bot) \\ \mathtt{f} & \text{if } \mathbf{E}a \in U(v_{x/a}(A_1(x)) = \mathtt{f}). \end{cases}$$

Let $\Gamma, \Delta, \Sigma, \Pi$ be sets of formulas. A multisequent δ is of form $\Gamma|\Delta|\Sigma|\Pi$. We say that δ is satisfied in (I, v), denoted by $I, v \models \Gamma|\Delta|\Sigma|\Pi$, if either $I, v \models \Gamma, I, v \models \Delta, I, v \models \Sigma$, or $I, v \models \Pi$, where

$I, v \models \Delta$ if for some formula $A \in \Delta, v(A) = \mathtt{t}$;
$I, v \models \Theta$ if for some formula $B \in \Theta, v(B) = \top$;
$I, v \models \Gamma$ if for some formula $C \in \Theta, v(C) = \bot$; and
$I, v \models \Pi$ if for some formula $D \in \Pi, v(D) = \mathtt{f}$.

A multisequent $\Gamma|\Delta|\Sigma|\Pi$ is valid, denoted by $\models \Gamma|\Delta|\Sigma|\Pi$, if for any interpretation I and assignment $v, I, v \models \Gamma|\Delta|\Sigma|\Pi$.

Proposition 3.1 Let $\Gamma, \Delta, \Sigma, \Pi$ be sets of atomic formulas. Then, $\Gamma|\Delta|\Sigma|\Pi$ is valid if and only if $\Gamma \cap \Delta \cap \Sigma \cap \Pi \neq \emptyset$.

Proof. Assume that $p \in \Gamma \cap \Delta \cap \Sigma \cap \Pi \neq \emptyset$. For any interpreation I and assignment v, if $v(p) = \mathtt{t}$ then $I, v \models \Gamma$; if $v(p) = \top$ then $I, v \models \Delta$; if $v(p) = \bot$ then $I, v \models \Sigma$; otherwise, $v(p) = \mathtt{f}, I, v \models \Pi$. Hence, $I, v \models \Gamma|\Delta|\Sigma|\Pi$.

Conversely, assume that $\Gamma \cap \Delta \cap \Sigma \cap \Pi = \emptyset$. Let U be the set of all the constants occurring in Γ, Δ, Σ or Π. Define I and v such that for any atomic formula l,

$$v(l) = \begin{cases} *_4 & \text{if } l = \Gamma_{*_1} \cap \Gamma_{*_2} \cap \Gamma_{*_3} \\ \star_3 & \text{otherwise, if } l \in \Gamma_{\star_1} \cap \Gamma_{\star_2} - \bigcup_{*_1,*_2,*_3} \Gamma_{*_1} \cap \Gamma_{*_2} \cap \Gamma_{*_3} \\ f_\neg(\#) & \text{otherwise, if } l \in \Gamma_\# - \bigcup_{\star_1,\star_2} \Gamma_{\star_1} \cap \Gamma_{\star_2} \end{cases}$$

where $\{*_1, *_2, *_3, *_4\} = \{\mathtt{t}, \top, \bot, \mathtt{f}\}$, and $\Gamma_{\mathtt{t}} = \Gamma, \Gamma_\top = \Delta, \Gamma_\bot = \Sigma, \Gamma_{\mathtt{f}} = \Pi$; $*_1 \neq *_2 \neq *_3 \neq \star_1; \star_1 \neq \star_2 \in \mathbf{B}_2^2$, and $\# \in \mathbf{B}_2^2$. Then, $I, v \not\models \Gamma|\Delta|\Sigma|\Pi$.

4. Gentzen deduction system

Gentzen deduction system $\mathbf{G}_2^2$ contains the following axioms and deduction rules:

• Axioms:

$$(\mathbf{A})\ \frac{\Pi \cap \Sigma \cap \Delta \cap \Gamma \neq \emptyset}{\Gamma|\Delta|\Sigma|\Pi},$$

where $\Gamma, \Delta, \Sigma, \Pi$ are sets of atomic formulas.

• Deduction rules for unary logical connective $\neg$:

$$(\neg^A)\ \frac{\Gamma|\Delta|\Sigma|A, \Pi}{\Gamma, \neg A|\Delta|\Sigma|\Pi} \quad (\neg^B)\ \frac{\Gamma|\Delta|\Sigma, B|\Pi}{\Gamma|\Delta, \neg B|\Sigma|\Pi}$$
$$(\neg^C)\ \frac{\Gamma|\Delta, C|\Sigma|\Pi}{\Gamma|\Delta|\Sigma, \neg C|\Pi} \quad (\neg^D)\ \frac{\Gamma, D|\Delta|\Sigma|\Pi}{\Gamma|\Delta|\Sigma|\neg D, \Pi}$$

• Deduction rules for binary logical connective $\wedge$:

$$(\wedge^A)\ \frac{\Gamma, A_1|\Delta|\Sigma|\Pi \quad \Gamma, A_2|\Delta|\Sigma|\Pi}{\Gamma, A_1 \wedge A_2|\Delta|\Sigma|\Pi}$$

$$(\wedge_1^B)\ \frac{\Gamma|\Delta, B_1|\Sigma|\Pi \quad \Gamma|\Delta, B_2|\Sigma|\Pi}{\Gamma|\Delta, B_1 \wedge B_2|\Sigma|\Pi}$$
$$(\wedge_2^B)\ \frac{\Gamma, B_1|\Delta|\Sigma|\Pi \quad \Gamma|\Delta, B_2|\Sigma|\Pi}{\Gamma|\Delta, B_1 \wedge B_2|\Sigma|\Pi}$$
$$(\wedge_3^B)\ \frac{\Gamma|\Delta, B_1|\Sigma|\Pi \quad \Gamma, B_2|\Delta|\Sigma|\Pi}{\Gamma|\Delta, B_1 \wedge B_2|\Sigma|\Pi}$$

$$(\wedge_1^C)\ \frac{\Gamma|\Delta|\Sigma, C_1|\Pi \quad \Gamma|\Delta|\Sigma, C_2|\Pi}{\Gamma|\Delta|\Sigma, C_1 \wedge C_2|\Pi}$$
$$(\wedge_2^C)\ \frac{\Gamma, C_1|\Delta|\Sigma|\Pi \quad \Gamma|\Delta|\Sigma, C_2|\Pi}{\Gamma|\Delta|\Sigma, C_1 \wedge C_2|\Pi}$$
$$(\wedge_3^C)\ \frac{\Gamma|\Delta|\Sigma, C_1|\Pi \quad \Gamma, C_2|\Delta|\Sigma|\Pi}{\Gamma|\Delta|\Sigma, C_1 \wedge C_2|\Pi}$$

$$(\wedge_1^D)\ \frac{\Gamma|\Delta|\Sigma|D_1, \Pi}{\Gamma|\Delta|\Sigma|D_1 \wedge D_2, \Pi}$$
$$(\wedge_2^D)\ \frac{\Gamma|\Delta|\Sigma|D_2, \Pi}{\Gamma|\Delta|\Sigma|D_1 \wedge D_2, \Pi}$$
$$(\wedge_3^D)\ \frac{\Gamma|\Delta, D_1|\Sigma|\Pi \quad \Gamma|\Delta|\Sigma, D_2, \Pi}{\Gamma|\Delta|\Sigma|D_1 \wedge D_2, \Pi}$$
$$(\wedge_4^D)\ \frac{\Gamma|\Delta|\Sigma, D_1|\Pi \quad \Gamma|\Delta, D_2|\Sigma|\Pi}{\Gamma|\Delta|\Sigma|D_1 \wedge D_2, \Pi}$$

• Deduction rules for binary logical connective $\vee$:

$$(\vee_1^A)\ \frac{\Gamma, A_1|\Delta|\Sigma|\Pi}{\Gamma, A_1 \vee A_2|\Delta|\Sigma|\Pi}$$
$$(\vee_2^A)\ \frac{\Gamma, A_2|\Delta|\Sigma|\Pi}{\Gamma, A_1 \vee A_2|\Delta|\Sigma|\Pi}$$
$$(\vee_3^A)\ \frac{\Gamma|\Delta|\Sigma, A_1|\Pi \quad \Gamma|\Delta, A_2|\Sigma|\Pi}{\Gamma, A_1 \vee A_2|\Delta|\Sigma|\Pi}$$
$$(\vee_4^A)\ \frac{\Gamma|\Delta, A_1|\Sigma|\Pi \quad \Gamma|\Delta|\Sigma, A_2|\Pi}{\Gamma, A_1 \vee A_2|\Delta|\Sigma|\Pi}$$

$$(\vee_1^B)\ \frac{\Gamma|\Delta, B_1|\Sigma|\Pi \quad \Gamma|\Delta, B_2|\Sigma|\Pi}{\Gamma|\Delta, B_1 \vee B_2|\Sigma|\Pi}$$
$$(\vee_2^B)\ \frac{\Gamma|\Delta|\Sigma|B_1, \Pi \quad \Gamma|\Delta, B_2|\Sigma|\Pi}{\Gamma|\Delta, B_1 \vee B_2|\Sigma|\Pi}$$
$$(\vee_3^B)\ \frac{\Gamma|\Delta, B_1|\Sigma|\Pi \quad \Gamma|\Delta|\Sigma|B_2, \Pi}{\Gamma|\Delta, B_1 \vee B_2|\Sigma|\Pi}$$

$$(\vee_1^C)\ \frac{\Gamma|\Delta|\Sigma, C_1|\Pi \quad \Gamma|\Delta|\Sigma, C_2|\Pi}{\Gamma|\Delta|\Sigma, C_1 \vee C_2|\Pi}$$
$$(\vee_2^C)\ \frac{\Gamma|\Delta|\Sigma|C_1, \Pi \quad \Gamma|\Delta|\Sigma, C_2|\Pi}{\Gamma|\Delta|\Sigma, C_1 \vee C_2|\Pi}$$
$$(\vee_3^C)\ \frac{\Gamma|\Delta|\Sigma, C_1|\Pi \quad \Gamma|\Delta|\Sigma|C_2, \Pi}{\Gamma|\Delta|\Sigma, C_1 \vee C_2|\Pi}$$

$$(\vee^D)\ \frac{\Gamma|\Delta|\Sigma|D_1, \Pi \quad \Gamma|\Delta|\Sigma|D_2, \Pi}{\Gamma|\Delta|\Sigma|D_1 \vee D_2, \Pi}$$

• Deduction rules for quantifier $\forall$:

$$(\forall^A)\ \frac{\Gamma, A(x)|\Delta|\Sigma|\Pi}{\Gamma, \forall x A(x)|\Delta|\Sigma|\Pi} \qquad (\forall^B_1)\ \frac{\Gamma|\Delta, B(t)|\Sigma|\Pi \ \ \Gamma, B(x)|\Delta|\Sigma|\Pi}{\Gamma|\forall x B(x), \Delta|\Sigma|\Pi}$$

$$(\forall^B_2)\ \frac{\Gamma|\Delta, B(t)|\Sigma|\Pi \ \ \Gamma|\Delta, B(x)|\Sigma|\Pi}{\Gamma|\forall x B(x), \Delta|\Sigma|\Pi}$$

$$(\forall^C_1)\ \frac{\Gamma|\Delta|\Sigma, C(t)|\Pi \ \ \Gamma, C(x)|\Delta|\Sigma|\Pi}{\Gamma|\forall x B(x), \Delta|\Sigma|\Pi} \qquad (\forall^D)\ \frac{\Gamma|\Delta|\Sigma|D(t), \Pi}{\Gamma|\Delta|\Sigma|\forall x D(x), \Pi}$$

$$(\forall^C_2)\ \frac{\Gamma|\Delta|\Sigma, C(t)|\Pi \ \ \Gamma|\Delta|\Sigma, C(x)|\Pi}{\Gamma|\Delta|\Sigma, \forall x C(x)|\Pi}$$

where t is a term and x is a new variable not occurring free in Γ, Δ, Σ and Π.

Definition 4.1 $\vdash\ \Gamma|\Delta|\Sigma|\Pi$ if there is a sequence $\{\Delta_1|\Theta_1|\Gamma_1|\Pi_1, ..., \Delta_n|\Theta_n|\Gamma_n|\Pi_n\}$ of multisequents such that $\Delta_n|\Theta_n|\Gamma_n|\Pi_n = \Gamma|\Delta|\Sigma|\Pi$, and for each $1 \leq i \leq n, \Delta_i|\Theta_i|\Gamma_i|\Pi_i$ is deduced from the previous multisequents by one of the deduction rules in $\mathbf{G}^2_2$.

Theorem 4.2 For any multisequent $\Gamma|\Delta|\Sigma|\Pi$, if $\models \Gamma|\Delta|\Sigma|\Pi$ then $\vdash \Gamma|\Delta|\Sigma|\Pi$.

Proof. We prove that axioms are valid and deduction rules preserve validity. Fix an interpretation I.

To verify the validity of the axiom, assume that $\Gamma \cap \Delta \cap \Sigma \cap \Pi \neq \emptyset$. Then, there is an atomic formula $l \in \Gamma \cap \Delta \cap \Sigma \cap \Pi$, and for any assignment v, if $v(l) = \mathtt{t}$ then $I, v \models \Gamma$; if $v(l) = \top$ then $I, v \models \Delta$, if $v(l) =\perp$ then $I, v \models \Sigma$, otherwise, $I, v \models \Pi$, and each of which implies $I, v \models \Gamma|\Delta|\Sigma|\Pi$.

To verify that $(\neg^B)$ preserves validity, assume that for any assignment $v, I, v \models \Gamma|\Delta|\Sigma, A|\Pi$. If $I, v \models \Gamma|\Delta|\Sigma|\Pi$ then $I, v \models \Gamma|\Delta, \neg A|\Sigma|\Pi$; otherwise, $v(A) =\perp$, and by the definition of $f_\neg, v(\neg A) = \top, I, v \models \Delta, \neg A$, and hence, $I, v \models \Gamma|\Delta, \neg A|\Sigma|\Pi$.

To verify that $(\wedge^D_3)$ preserves validity, assume that for any assignment v,

$$v \models \Gamma|\Delta, D_1|\Sigma|\Pi,$$
$$v \models \Gamma|\Delta|\Sigma, D_2|\Pi.$$

For any assignment v, if $v \models \Gamma|\Delta|\Sigma|\Pi$ then $v \models \Gamma|\Delta|\Sigma|D_1 \wedge D_2, \Pi$; otherwise, $v(D_1) = \top, v(D_2) =\perp$, and by the definition of $\cap, v(D_1 \wedge D_2) = \mathtt{f}$, $v \models D_1 \wedge D_2, \Pi$, and hence, $v \models \Gamma|\Delta|\Sigma|B_1 \wedge B_2, \Pi$.

To verify that $(\vee^A_4)$ preserves validity, assume that for any assignment v,

$$v \models \Gamma|\Delta, A_1|\Sigma|\Pi,$$
$$v \models \Gamma|\Delta|\Sigma, A_2|\Pi.$$

For any assignment v, if $v \models \Gamma|\Delta|\Sigma|\Pi$ then $v \models \Gamma, A_1 \vee A_2|\Delta|\Sigma|\Pi$; otherwise, $v(A_1) = \top, v(A_2) =\perp$, and by the definition of $\cup, v(A_1 \vee A_2) = \mathtt{t}$, $v \models A_1 \vee A_2, \Gamma$, and hence, $v \models \Gamma, A_1 \vee A_2|\Delta|\Sigma|\Pi$.

To verify that $(\forall^B_1)$ preserves the validity, assume that for any assignment v, $I, v \models \Gamma|\Delta, B(t)|\Sigma|\Pi\Gamma$ and $I, v \models \Gamma, B(x)|\Delta|\Sigma|\Pi$. For any assignment v, if $I, v \models \Gamma|\Delta|\Sigma|\Pi$ then $I, v \models \Gamma|\Delta, \forall x B(x)|\Sigma|\Pi$; otherwise, by induction assumption, $v(B(t)) = \top$ or $v(B(x)) = \mathtt{t}$, i.e., for any $a \in U$, either $v(B(t)) = \top$ or $v_{x/a}(B(x)) = \mathtt{t}$, i.e., $v(\forall x B(x)) = \top$.

To verify that $(\forall^B_2)$ preserves the validity, assume that for any assignment v, $I, v \models \Gamma|\Delta, B(t)|\Sigma|\Pi\Gamma$ and $I, v \models \Gamma|\Delta, B(x)|\Sigma|\Pi$. For any assignment v, if $I, v \models \Gamma|\Delta|\Sigma|\Pi$ then $I, v \models \Gamma|\Delta, \forall x B(x)|\Sigma|\Pi$; otherwise, by induction assumption, $v(B(t)) = \top$ or $v(B(x)) = \top$, i.e., for any $a \in U$, either $v(B(t)) = \top$ or $v_{x/a}(B(x)) = \top$, i.e., $v(\forall x B(x)) = \top$.

Similar for other deduction rules.

Theorem 4.3 For any multisequent $\Gamma|\Delta|\Sigma|\Pi$, if $\models \Gamma|\Delta|\Sigma|\Pi$ then $\vdash \Gamma|\Delta|\Sigma|\Pi$.

Proof. Given a multisequent $\Gamma|\Delta|\Sigma|\Pi$, we construct a tree T such that either

(i) for each branch ξ of T, each multisequent $\Gamma'|\Delta'|\Sigma'|\Pi'$ at the leaf of ξ is an axiom, or

(ii) there is an assignment v such that $v \not\models \Gamma|\Delta|\Sigma|\Pi$.

T is constructed as follows:

• the root of T is $\Gamma|\Delta|\Sigma|\Pi$;

• for a node ξ, if for each sequent $\Gamma'|\Delta'|\Sigma'|\Pi'$ at $\xi, \Gamma' \cup \Delta' \cup \Sigma' \cup \Pi'$ is a set of atomic formulas then the node is a leaf;

• otherwise, ξ has the direct child node containing the following multisequents:

$$\begin{cases} \Gamma_1|\Delta_1|\Sigma_1|A, \Pi_1 & \text{if } \Gamma_1, \neg A|\Delta_1|\Sigma_1|\Pi_1 \in \xi \\ \Gamma_1|\Delta_1|\Sigma_1, B|\Pi_1 & \text{if } \Gamma_1|\Delta_1, \neg B|\Sigma_1|\Pi_1 \in \xi \\ \Gamma_1|\Delta_1|\Sigma_1, C|\Pi_1 & \text{if } \Gamma_1|\Delta_1|\Sigma_1, \neg C|\Pi_1 \in \xi \\ \Gamma_1, D|\Delta_1|\Sigma_1|\Pi_1 & \text{if } \Gamma_1|\Delta_1|\Sigma_1|\neg D, \Pi_1 \in \xi \end{cases}$$

and

$$\left\{\begin{array}{ll}
\left[\begin{array}{l}\Gamma_1, A_1|\Delta_1|\Sigma_1|\Pi_1\\ \Gamma_1, A_2|\Delta_1|\Sigma_1|\Pi_1\end{array}\right. & \text{if } \Gamma_1, A_1 \wedge A_2|\Delta_1|\Sigma_1|\Pi_1 \in \xi\\
\left\{\begin{array}{l}
\left[\begin{array}{l}\Gamma_1|\Delta_1, B_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1, B_2|\Sigma_1|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1, B_1|\Delta_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1, B_2|\Sigma_1|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1, B_1|\Sigma_1|\Pi_1\\ \Gamma_1, B_2|\Delta_1|\Sigma_1|\Pi_1\end{array}\right.
\end{array}\right. & \text{if } \Gamma_1|\Delta_1, B_1 \wedge B_2|\Sigma_1|\Pi_1 \in \xi\\
\left\{\begin{array}{l}
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, C_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1, C_2|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1, C_1|\Delta_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1, C_2|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, C_1|\Pi_1\\ \Gamma_1, C_2|\Delta_1|\Sigma_1|\Pi_1\end{array}\right.
\end{array}\right. & \text{if } \Gamma_1|\Delta_1|\Sigma_1, C_1 \wedge C_2|\Pi_1 \in \xi\\
\left\{\begin{array}{l}
\Gamma_1|\Delta_1|\Sigma_1|D_1, \Pi_1\\
\Gamma_1|\Delta_1|\Sigma_1|D_2, \Pi_1\\
\left[\begin{array}{l}\Gamma_1|\Delta_1, D_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1, D_2|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, D_1|\Pi_1\\ \Gamma_1|\Delta_1, D_2|\Sigma_1|\Pi_1\end{array}\right.
\end{array}\right. & \text{if } \Gamma_1|\Delta_1|\Sigma_1|D_1 \wedge D_2, \Pi_1 \in \xi
\end{array}\right.$$

and

$$\left\{\begin{array}{ll}
\left\{\begin{array}{l}
\Gamma_1, A_1|\Delta_1|\Sigma_1|\Pi_1\\
\Gamma_1, A_2|\Delta_1|\Sigma_1|\Pi_1\\
\left[\begin{array}{l}\Gamma_1|\Delta_1, A_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1, A_2|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, A_1|\Pi_1\\ \Gamma_1|\Delta_1, A_2|\Sigma_1|\Pi_1\end{array}\right.
\end{array}\right. & \text{if } \Gamma_1, A_1 \vee A_2|\Delta_1|\Sigma_1|\Pi_1 \in \xi\\
\left\{\begin{array}{l}
\left[\begin{array}{l}\Gamma_1|\Delta_1, B_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1, B_2|\Sigma_1|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1, B_1|\Sigma_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1|\Pi_1, B_2\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1|\Pi_1, B_1\\ \Gamma_1|\Delta_1, B_2|\Sigma_1|\Pi_1\end{array}\right.
\end{array}\right. & \text{if } \Gamma_1|\Delta_1, B_1 \vee B_2|\Sigma_1|\Pi_1 \in \xi\\
\left\{\begin{array}{l}
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, C_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1, C_2|\Pi_1\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, C_1|\Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1|\Pi_1, C_2\end{array}\right.\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1|\Pi_1, C_1\\ \Gamma_1|\Delta_1|\Sigma_1, C_2|\Pi_1\end{array}\right.
\end{array}\right. & \text{if } \Gamma_1|\Delta_1|\Sigma_1, C_1 \vee C_2|\Pi_1 \in \xi\\
\left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1|D_1, \Pi_1\\ \Gamma_1|\Delta_1|\Sigma_1|D_2, \Pi_1\end{array}\right. & \text{if } \Gamma_1|\Delta_1|\Sigma_1|D_1 \vee D_2, \Pi_1 \in \xi
\end{array}\right.$$

and

$$\begin{cases} \left[\begin{array}{l}\Gamma_1, A_1(c)|\Delta_1|\Sigma_1|\Pi_1 \\ c \text{ does not occur in current } T\end{array}\right. & \text{if } \Gamma_1, \forall x A_1(x)|\Delta_1|\Sigma_1|\Pi_1 \in \xi \\ \begin{cases} \left[\begin{array}{l}\Gamma_1|\Delta_1, B_1(t)|\Sigma_1|\Pi_1 \\ \Gamma_1, B_1(c)|\Delta_1|\Sigma_1|\Pi_1 \\ c \text{ does not occur in current } T\end{array}\right. \\ \left[\begin{array}{l}\Gamma_1|\Delta_1, B_1(t)|\Sigma_1|\Pi_1 \\ \Gamma_1|\Delta_1, B_1(c)|\Sigma_1|\Pi_1 \\ c \text{ does not occur in current } T\end{array}\right. \end{cases} & \text{if } \Gamma_1|\Delta_1, \forall x B_1(x)|\Sigma_1|\Pi_1 \in \xi \\ \begin{cases} \left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, C_1(t)|\Pi_1 \\ \Gamma_1, C_1(c)|\Delta_1|\Sigma_1|\Pi_1 \\ c \text{ does not occur in current } T\end{array}\right. \\ \left[\begin{array}{l}\Gamma_1|\Delta_1|\Sigma_1, C_1(t)|\Pi_1 \\ \Gamma_1|\Delta_1|\Sigma_1, C_1(c)|\Pi_1 \\ c \text{ does not occur in current } T\end{array}\right. \end{cases} & \text{if } \Gamma_1|\Delta_1|\Sigma_1, \forall x C_1(x)|\Pi_1 \in \xi \\ \Gamma_1|\Delta_1|\Sigma_1|D_1(t), \Pi_1 & \text{if } \Gamma_1|\Delta_1|\Sigma_1|\forall x D_1(x), \Pi_1 \in \xi \end{cases}$$

and

- for each $\Gamma_2|\Delta_2, B'_1(t)|\Sigma_2|\Pi_2 \in T$ such that $\Gamma_2|\Delta_2, B'_1(t)|\Sigma_2|\Pi_2$ has not be applied to a constant c, and for each child node $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ of $\Gamma_2|\Delta_2, B'_1(t)|\Sigma_2|\Pi_2$, let the child node of $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ contain sequent $\Gamma_3|\Delta_3, B'_1(c)|\Sigma_3|\Pi_3$;
- for each $\Gamma_2|\Delta_2|\Sigma_2, C'_1(t)|\Pi_2 \in T$ such that $\Gamma_2|\Delta_2|\Sigma_2, C'_1(t)|\Pi_2$ has not be applied to a constant c, and for each child node $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ of $\Gamma_2|\Delta_2|\Sigma_2, C'_1(t)|\Pi_2$, let the child node of $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ contain sequent $\Gamma_3|\Delta_3|\Sigma_3, C'_1(c)|\Pi_3$;
- for each $\Gamma_2|\Delta_2|\Sigma_2|D'_1(t), \Pi_2 \in T$ such that $\Gamma_2|\Delta_2|\Sigma_2|D'_1(t), \Pi_2$ has not be applied to a constant c, and for each child node $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ of $\Gamma_2|\Delta_2|\Sigma_2|D'_1(t), \Pi_2$, let the child node of $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ contain sequent $\Gamma_3|\Delta_3|\Sigma_3|D'_1(c), \Pi_3$,

where $\left[\begin{array}{l}\delta_1 \\ \delta_2\end{array}\right.$ represents that δ_1, δ_2 are at a same child node; and $\begin{cases}\delta_1 \\ \delta_2\end{cases}$ represents that δ_1, δ_2 are at different direct children nodes.

Lemma 4.4 If there is a branch $\xi \subseteq T$ such that each multisequent $\Gamma'|\Delta'|\Sigma'|\Pi' \in \xi$ is an axiom in $\mathbf{G}^2_2$ then ξ is a proof of $\Gamma|\Delta|\Sigma|\Pi$.

Proof. By the definition of T, T is a proof tree of $\Gamma|\Delta|\Sigma|\Pi$.

Lemma 4.5 For each branch $\xi \subseteq T$, there is a multisequent $\Gamma'|\Delta'|\Sigma'|\Pi' \in \xi$ is not an axiom in $\mathbf{G}^2_2$ then there is an assignment v such that $v \not\models \Gamma|\Delta|\Sigma|\Pi$.

Proof. Let γ be the set of all the atomic multisequents in T which is not an axiom.

Let

$$\begin{array}{l}\mathbf{A} = \{l : l \in \Gamma', \Gamma'|\Delta'|\Sigma'|\Pi' \in \gamma\}, \\ \mathbf{B} = \{l : l \in \Delta', \Gamma'|\Delta'|\Sigma'|\Pi' \in \gamma\}, \\ \mathbf{C} = \{l : l \in \Sigma', \Gamma'|\Delta'|\Sigma'|\Pi' \in \gamma\}, \\ \mathbf{D} = \{l : l \in \Pi', \Gamma'|\Delta'|\Sigma'|\Pi' \in \gamma\},\end{array}$$

and U be the set of all the constants occurring in $\mathbf{A} \cup \mathbf{B} \cup \mathbf{C} \cup \mathbf{D}$. Define an interpretation I such that

$$I(p(c_1, ..., c_n)) = \begin{cases} \mathtt{f} & \text{if } p(c_1, ..., c_n) \in \mathbf{A} \\ \bot & \text{if } p(c_1, ..., c_n) \in \mathbf{B} \\ \top & \text{if } p(c_1, ..., c_n) \in \mathbf{C} \\ \mathtt{t} & \text{if } p(c_1, ..., c_n) \in \mathbf{D} \\ \mathtt{t} & \text{otherwise.} \end{cases}$$

We proved by induction on tree that each $\xi \in T$ contains a multisequent $\Gamma'|\Delta'|\Sigma'|\Pi' \in \xi$ which is not satisfied by v.

Case 1. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'', \neg B|\Sigma'|\Pi' \in \xi$. Then, ξ has a direct child node containing $\Gamma'|\Delta''|\Sigma', B|\Pi'$. By induction assumption, if $v \not\models \Gamma'|\Delta''|\Sigma', B|\Pi'$, i.e., $v(B) \neq \bot$, then $v \not\models \Gamma'|\Delta'', \neg B|\Sigma'|\Pi'$.

Case 2. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'', A_1 \wedge A_2|\Delta'|\Sigma'|\Pi' \in \xi$. Then, ξ has a direct child node containing $\Gamma'', A_1|\Delta'|\Sigma'|\Pi'$ and $\Delta'', A_2|\Theta'|\Gamma'$. By induction assumption, if either $v \not\models \Gamma'', A_1|\Delta'|\Sigma'|\Pi'$, or $v \not\models \Gamma'', A_2|\Delta'|\Sigma'|\Pi'$ then $v \not\models \Gamma'', A_1 \wedge A_2|\Delta'|\Sigma'|\Pi'$.

Case 3. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'', B_1 \wedge B_2|\Sigma''|\Pi' \in \xi$. Then, ξ has three direct children node containing

$$\begin{array}{l}\Gamma'|\Delta'', B_1|\Sigma'|\Pi', \Gamma'|\Delta'', B_2|\Sigma'|\Pi'; \\ \Gamma'|\Delta'', B_1|\Sigma'|\Pi', \Gamma'|\Delta''|\Sigma'|\Pi', B_2; \\ \Gamma'|\Delta''|\Sigma'|\Pi', B_1, \Gamma'|\Delta'', B_2|\Sigma'|\Pi';\end{array}$$

respectively. By induction assumption, if

either $v \not\models \Gamma'|\Delta'', B_1|\Sigma'|\Pi'$, or $v \not\models \Gamma'|\Delta'', B_2|\Sigma'|\Pi'$;
either $v \not\models \Gamma'|\Delta'', B_1|\Sigma'|\Pi'$, or $v \not\models \Gamma'|\Delta''|\Sigma'|B_2, \Pi'$;
either $v \not\models \Gamma'|\Delta''|\Sigma'|B_1, \Pi'$, or $v \not\models \Gamma'|\Delta'', B_2|\Sigma'|\Pi'$;

then $v \not\models \Gamma'|\Delta'', B_1 \wedge B_2|\Sigma'|\Pi'$.

Case 4. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'', C_1 \wedge C_2|\Pi' \in \xi$. Then, ξ has three direct children node containing

$$\Gamma'|\Delta'|\Sigma'', C_1|\Pi', \Gamma'|\Delta'|\Sigma'', C_2|\Pi';$$
$$\Gamma'|\Delta'|\Sigma'', C_1|\Pi', \Gamma'|\Delta'|\Sigma''|\Pi', C_2;$$
$$\Gamma'|\Delta'|\Sigma''|\Pi', C_1, \Gamma'|\Delta'|\Sigma'', C_2|\Pi';$$

respectively. By induction assumption, if

either $v \not\models \Gamma'|\Delta'|\Sigma'', C_1|\Pi'$, or $v \not\models \Gamma'|\Delta'|\Sigma'', C_2|\Pi'$;
either $v \not\models \Gamma'|\Delta'|\Sigma'', C_1|\Pi'$, or $v \not\models \Gamma'|\Delta'|\Sigma''|C_2, \Pi'$;
either $v \not\models \Gamma'|\Delta'|\Sigma''|C_1, \Pi'$, or $v \not\models \Gamma'|\Delta'|\Sigma'', C_2|\Pi'$;

then $v \not\models \Gamma'|\Delta'|\Sigma'', C_1 \wedge C_2|\Pi'$.

Case 5. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'|D_1 \wedge D_2, \Pi'' \in \xi$. Then, ξ has four direct children nodes containing one of the following multisequents:

$$\Gamma'|\Delta'|\Sigma'|D_1, \Pi'',$$
$$\Gamma'|\Delta'|\Sigma'|D_2, \Pi'',$$
$$\Gamma'|\Delta', D_1|\Sigma'|\Pi''; \Gamma'|\Delta'|\Sigma', D_2|\Pi'',$$
$$\Gamma'|\Delta'|\Sigma', D_1|\Pi''; \Gamma'|\Delta', D_2|\Sigma'|\Pi''$$

By induction assumption,

$v \not\models \Gamma'|\Delta'|\Sigma'|D_1, \Pi''$,
$v \not\models \Gamma'|\Delta'|\Sigma'|D_2, \Pi''$,
$v \not\models \Gamma'|\Delta', D_1|\Sigma'|\Pi''$; or $v \not\models \Gamma'|\Delta'|\Sigma', D_2|\Pi''$,
$v \not\models \Gamma'|\Delta'|\Sigma', D_1|\Pi''$; or $v \not\models \Gamma'|\Delta', D_2|\Sigma'|\Pi''$

Hence, $v \not\models \Gamma'|\Delta'|\Sigma'|D_1 \wedge D_2, \Pi''$.

Case 6. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'', \forall x A_1(x)|\Delta'|\Sigma'|\Pi' \in \xi$. Then, ξ has a direct child node containing $\Gamma'', A_1(d)|\Delta'|\Sigma'|\Pi'$ for each constant d occurring in ξ. By induction assumption, $v \not\models \Gamma'', A_1(d)|\Delta'|\Sigma'|\Pi'$ for some d occurring in ξ, i.e., $v \not\models \Gamma'', \forall x A_1(x)|\Delta'|\Sigma'|\Pi'$.

Case 7. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'', \forall x B_1(x)|\Sigma''|\Pi' \in \xi$. Then, ξ has two direct child nodes containing either

$$\Gamma'|\Delta'', B_1(c)|\Sigma'|\Pi', \Gamma', B_1(d)|\Delta''|\Sigma'|\Pi'$$

for each d occurring in ξ, or

$$\Gamma'|\Delta'', B_1(c)|\Sigma'|\Pi', \Gamma'|\Delta'', B_1(d)|\Sigma'|\Pi'$$

for each d occurring in ξ. By induction assumption, (1) either $v \not\models \Gamma'|\Delta'', B_1(c)|\Sigma'|\Pi'$, or $v \not\models \Gamma', B_1(d)|\Delta''|\Sigma'|\Pi'$ for some d; and (2) either $v \not\models \Gamma'|\Delta'', B_1(c)|\Sigma'|\Pi'$, or $v \not\models \Gamma'|\Delta'', B_1(d)|\Sigma'|\Pi'$ for some d. Then $v \not\models \Gamma'|\Delta'', \forall x B_1(x)|\Sigma'|\Pi'$.

Case 8. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'', \forall x C_1(x)|\Pi' \in \xi$. Then, ξ has two direct children nodes containing either

$$\Gamma'|\Delta''|\Sigma', C_1(c)|\Pi', \Gamma', C_1(d)|\Delta''|\Sigma'|\Pi'$$

for each d occurring in ξ, or

$$\Gamma'|\Delta''|\Sigma', C_1(c)|\Pi', \Gamma'|\Delta''|\Sigma', C_1(d)|\Pi'$$

for each d occurring in ξ. By induction assumption, (1) either $v \not\models \Gamma'|\Delta''|\Sigma', C_1(c)|\Pi'$, or $v \not\models \Gamma', C_1(d)|\Delta''|\Sigma'|\Pi'$ for some d; and (2) either $v \not\models \Gamma'|\Delta''|\Sigma', C_1(c)|\Pi'$, or $v \not\models \Gamma'|\Delta''|\Sigma', C_1(d)|\Pi'$ for some d. Then $v \not\models \Gamma'|\Delta''|\Sigma', \forall x C_1(x)|\Pi'$.

Case 9. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'|\forall x D_1(x), \Pi'' \in \xi$. Then, ξ has a direct child node containing $\Gamma|\Delta|\Sigma|D_1(c), \Pi$. By induction assumption, $v \not\models \Gamma'|\Delta'|\Sigma'|D_1(c), \Pi''$, and hence, $v \not\models \Gamma'|\Delta'|\Sigma'|\forall x D_1(x), \Pi''$.

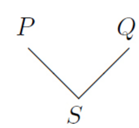
Similar for other cases.

5. Discussion

In the proof of completeness theorem, given a sequent $\Gamma \Rightarrow \Delta$ to be proved and a deduction rule of form

$$\frac{P \qquad Q}{S}$$

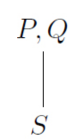
where P, Q, S are sequents, we decompose a node containing S into two children nodes containg P and Q, respectively:



Given a deduction rule of form

$$\left[\begin{array}{l} \frac{P}{S} \\ \frac{Q}{S} \end{array}\right.$$

we merge sequents P and Q into one sequent P, Q:

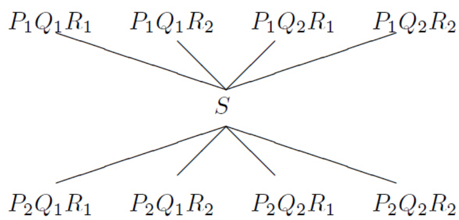


In the end, we get a tree T such that each sequent at the leaf node of T is atomic. If each leaf has an axiom then T is a proof tree; otherwise, there is a branch γ of T such that the leaf node of γ contains no axiom. Then, we define an assignment v in which the sequent $\Gamma \Rightarrow \Delta$ is not satisfied.

For multisequents, a node containing S which has three deduction rules

$$\left\{\begin{array}{l}\dfrac{P_1 \quad P_2}{S}\\ \dfrac{Q_1 \quad Q_2}{S}\\ \dfrac{R_1 \quad R_2}{S}\end{array}\right.$$

has eight children nodes:



In another way, given a deduction rule

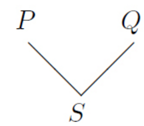
$$\frac{P \qquad Q}{S}$$

we merge sequents P and Q into one sequent P, Q:

Given a deduction rule:

$$\left\{\begin{array}{l}\dfrac{P}{S}\\ \dfrac{Q}{S}\end{array}\right.$$

a node containing S has two children nodes containing P and Q, respectively:

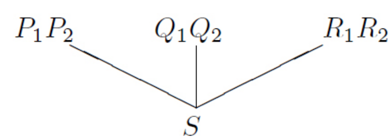


In the end, we get a tree T such that each sequent at the leaf node of T is atomic. If there is a branch γ such that each sequent at the leaf node of γ is an axiom then γ is a proof of $\Gamma \Rightarrow \Delta$; otherwise, for each leaf node ξ of T, there is a sequent at ξ is not an axiom. Then, we define an assignment v in which the sequent $\Gamma \Rightarrow \Delta$ is not satisfied.

For multisequents, a node containing S which has three deduction rules

$$\left\{\begin{array}{l}\dfrac{P_1 \quad P_2}{S}\\ \dfrac{Q_1 \quad Q_2}{S}\\ \dfrac{R_1 \quad R_2}{S}\end{array}\right.$$

has three children nodes:



Dually, for existential quantifier $\exists$ we have the following definition of truth-value:

$$v(\exists x A_1(x)) = \begin{cases} \mathtt{t} & \text{if } \mathbf{E}a \in U(v_{x/a}(A_1(x)) = \mathtt{t}) \\ \top & \text{if } \mathbf{A}a \in U(v_{x/a}(A_1(x)) \in \{\mathtt{f}, \top\})\&\mathbf{E}a \in U(v_{x/a}(A_1(x)) = \top) \\ \bot & \text{if } \mathbf{A}a \in U(v_{x/a}(A_1(x)) \in \{\mathtt{f}, \bot\})\&\mathbf{E}a \in U(v_{x/a}(A_1(x)) = \bot) \\ \mathtt{f} & \text{if } \mathbf{A}a \in U(v_{x/a}(A_1(x)) = \mathtt{f}). \end{cases}$$

and deduction rules:

$$(\exists^A)\ \frac{\Gamma, A(t)|\Delta|\Sigma|\Pi}{\Gamma, \exists x A(x)|\Delta|\Sigma|\Pi}$$

$$(\exists^B_1)\ \frac{\Gamma|\Delta, B(t)|\Sigma|\Pi \quad \Gamma|\Delta|\Sigma|B(x), \Pi}{\Gamma|\exists x B(x), \Delta|\Sigma|\Pi}$$

$$(\exists^B_2)\ \frac{\Gamma|\Delta, B(t)|\Sigma|\Pi \quad \Gamma|\Delta, B(x)|\Sigma|\Pi}{\Gamma|\exists x B(x), \Delta|\Sigma|\Pi}$$

$$(\exists^C_1)\ \frac{\Gamma|\Delta|\Sigma, C(t)|\Pi \quad \Gamma|\Delta|\Sigma|C(x), \Pi}{\Gamma|\exists x C(x), \Delta|\Sigma|\Pi}$$

$$(\exists^C_2)\ \frac{\Gamma|\Delta|\Sigma, C(t)|\Pi \quad \Gamma|\Delta|\Sigma, C(x)|\Pi}{\Gamma|\Delta|\Sigma, \exists x C(x)|\Pi}$$

$$(\exists^D)\ \frac{\Gamma|\Delta|\Sigma|D(x), \Pi}{\Gamma|\Delta|\Sigma|\exists x D(x), \Pi}$$

where t is a term and x is a new variable not occurring free in Γ, Δ, Σ and Π.

6. Conclusion

A Gentzen deduction system for $\mathbf{B}_2^2$-valued first-order logic is given and soundness and completeness theorems are proved.

A future work will consider different choices for defining the truth-values of quantified formulas. One choice is as follows:

• $\forall xA(x)$ has truth-value $\mathtt{t}$ if for each element $a, A(x/a)$ has truth-value $\mathtt{t}$;

• $\forall xA(x)$ has truth-value $\top$ if for each element $a, A(x/a)$ has truth-value $\top$;

• $\forall xA(x)$ has truth-value $\bot$ if for each element $a, A(x/a)$ has truth-value $\bot$;

• $\forall xA(x)$ has truth-value $\mathtt{f}$ if for each element $a, A(x/a)$ has truth-value $\mathtt{f}$.

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