ORIGINAL RESEARCH

A novel application of intuitionistic fuzzy sets theory in medical science: Bacillus colonies recognition

Hoda Davarzani[1], **Mohammadreza Amiri Khorheh**[2]

1. Department of Industrial Management and Logistics, Faculty of Engineering, Lund University, Sweden. 2. Resalat Medical Laboratory, Tehran, Iran

Correspondence: Hoda Davarzani. Address: Department of Industrial Management and Logistics, Faculty of Engineering, Lund University, 221 00 Lund, Sweden. Telephone: 0046-46-222-8649. Email: hoda.davarzani@tlog.lth.se.

Received: May 21, 2012 **Accepted:** July 5, 2012 **Online Published:** February 18, 2013
DOI: 10.5430/air.v2n2p1 **URL:** http://dx.doi.org/10.5430/air.v2n2p1

Abstract

This paper addresses four new distance measures as tools in pattern recognition for intuitionistic fuzzy sets (IFSs). Following the discussion on literature of distance measures, proposed measurement scales and the proof of their properties would be presented. In order to show the reliability of addressed formulations, this paper employs them in a part of medical diagnosis progress in bacillus colonies identification. The performed experiments test validity and reliability of the proposed models by running pattern recognition process on sixty cases of four different bacilli. Resulted outcomes by IFSs approach are compared with similar measures in regular fuzzy. Numerical comparisons reveal effectiveness of the proposed distance measurement scales and related pattern recognition progress.

Key words

Intuitionistic fuzzy sets, Distance measure, Medical diagnosis, Bacillus recognition

1 Introduction

Fuzzy sets were introduced in 1965 by Zadeh as a novel way of representing vagueness. This theory provides a tool to describe the characteristics of a too complex or ill-defined system to admit precise mathematical analysis. Most of the time, the logic of human reasoning is not based on traditional two-valued or even multi-valued logic, but logic with fuzzy truths. One of the applications of fuzzy thought is in pattern recognition. "A pattern is defined by the common denominator among the multiple instances of an entity" [1]. Pattern recognition problems contain the classification of an unknown pattern L given a set of K prototypes P_k , $k \in \{1,2,...,K\}$. Each prototype P_k belongs to a given class C_m, $m \in \{1,2,...,M\}$, which is specified by the indicator function A_k; and A_k will be equal to C_m if P_k belongs to the m^{th} class of C_m [2].

The techniques of pattern recognition are usually applied for situations which are inherently vague and uncertain [2-4]. Such situations arise when the information regarding the prototypes is "linguistic" and is based on the opinions and judgments of human experts. Examples of such situations are: handwritten character recognition, fingerprint recognition, human face recognition, classification of X-ray images, medical diagnosis and classification of remotely sensed data.

Research on the application of fuzzy set theory in supervised pattern recognition was started in 1966 by Bellman et al. The significance of fuzzy set theory in the area of pattern recognition, is adequately justified in [5] as:

- Representing linguistically phrased input features for processing.
- Providing an estimate (representation) of missing information in terms of membership values.
- Representing multiclass membership of ambiguous patterns and in generating rules and inferences in linguistic form.
- Extracting ill-defined image regions, primitives, and properties and describing relations among them as fuzzy subsets.

Medical diagnosis can be an example of pattern recognition. In most of medical diagnosis problems there exists a base pattern and experts make decision based on the similarity between the case and patterns [4, 6-11]. There are numerous visual diagnoses patterns in medical laboratory; where the diagnosis progress could be mentioned as pattern recognition.

Medical diagnosis is inherently combined with uncertainty, so fuzzy approach can be appropriate for it and also for diagnosis in laboratories [10, 12]; on the other hand, this uncertainty is not only in judgment but also in identification. Hence, there is a need for a kind of fuzzy approach to support the latter uncertainty. This could be type II fuzzy sets [13, 14], interval valued fuzzy [15, 16] or intuitionistic fuzzy sets [17-20]. Each of the mentioned models has its own advantages, for example the first two model (type II fuzzy sets and interval valued fuzzy sets) demonstrate the uncertainty in membership degree, applying membership degree for the first membership and with an interval for membership degree respectively; but the intuitionistic fuzzy sets (IFSs) reveals the uncertainty in membership degree via a non-membership degree. Intuitionistic fuzzy is the appropriate choice when exhibiting the non-membership degree is simpler than membership degree [21].

The main advantages of using a IFS framework similar to type II [2] are:

- By using IFSs we transform a vague pattern classification problem into a precise, well-defined, optimization problem.
- IFSs, unlike ordinary fuzzy sets, retain a controlled degree of uncertainty.

And, the main disadvantage of using a IFSs formulation is the relatively high computational complexity.

In medical laboratory the non-membership degree is a significant concept which can be used in formulation of pattern recognition progress of some circumstances. This paper concentrates on the formulation of this problem as an application of proposed measures.

This article is organized as follows: In Section 2 the concept of intuitionistic fuzzy sets and the researches about pattern recognition in this kind of fuzzy formulation are discussed. In Section 3 four new IFS distance measures are described which are specifically designed to measure the incompatibility of two intuitionistic fuzzy sets. In Section 4 the proposed measures are illustrated in a practical example of medical diagnosis in laboratory. Section 5 analyzes the results obtained from proposed measures and compares them with the results of regular fuzzy approach. Finally, the paper concludes with a brief summary in Section 6.

2 Theoretical background

In fuzzy set theory, the membership of an element to a fuzzy set is a single value between zero and one. But in reality, it may not always be certain that the degree of non-membership of an element in a fuzzy set is just equal to 1 minus the degree of membership. That is to say, there may be some hesitation degree. So, as a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets (IFSs) was introduced by Atanassov in 1983, which contains a non-membership degree beside membership degree [22]. Both degrees of membership and non-membership belong to the interval [0, 1], and their sum should not exceed 1. Formally, an IFS A in a universe X was defined as an object of the form:

$$A=\{(x,\mu_A(x),v_A(x))\,|\,x\in X\} \tag{1}$$

where $\mu_A(x)$ is called "the degree of membership of x in A" and $v_A(x)$, "the degree of non-membership of x in A", and

$$\pi_A(x)=1-\mu_A(x)-v_A(x) \tag{2}$$

is called "the hesitation degree of the element x to A". The class of IFSs in a universe X is denoted by IFS(X) [23].

Bustince and Burillo [24] showed that this notion coincides with the notion of vague sets proposed by Gau and Buehere in 1994. But in [25] the differences between vague sets and intuitionistic fuzzy sets have been shown. As important contents in fuzzy mathematics, similarity measure and distance measure between IFSs have attracted many researchers, which could be used as a tool in pattern recognition. Li and Cheng proposed similarity measures of IFSs and applied these measures to pattern recognition [26]. But Liang and Shi [27], as well as Mitchell [28] pointed out that Li and Cheng's measures [26] are not always effective in some cases, and made some modifications. Also, Szmidt and Kacprzyk [29] proposed four distance measures between IFSs, which were in some extent based on the geometric interpretation of intuitionistic fuzzy sets, and have some good geometric properties. But they are also not effective in some cases.

As mentioned earlier, these measures can be used as a tool in pattern recognition. From this point of view, pattern recognition researches in IFS could be classified into three groups:

- Using the distance concept and assigning each case to the nearest pattern [29-31];
- Using the similarity concept, introduced as duality of distance measure [22, 25-28, 32-34];
- Using cross-entropy concept [12].

Many measures of distance measures between intuitionistic fuzzy sets have been proposed by researches in recent years [12, 22, 25-36]. But one of the earliest ones has been developed by Szmidt and Kacprzyk [29]:

Let $A=\{(x,\mu_A(x),v_A(x))\,|\,x\in X\}$, $B=\{(x,\mu_B(x),v_B(x))\,|\,x\in X\}$ be two IFSs in $X=\{x_1, x_2, \ldots, x_n\}$. Based on the geometric interpretation of IFS, Szmidt and Kacprzyk [29] proposed the following four distance measures between A and B:

- Hamming distance:

$$d^1_{IFS}(A,B)=\frac{1}{2}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|v_A(x_i)-v_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|) \tag{3}$$

- Euclidean distance:

$$e^1_{IFS}(A,B)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2+(v_A(x_i)-v_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2} \tag{4}$$

- Normalized Hamming distance:

$$l^1_{IFS}(A,B)=\frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|v_A(x_i)-v_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|) \tag{5}$$

- Normalized Euclidean distance:

$$q^{1}_{IFS}(A,B)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2+(v_A(x_i)-v_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2} \quad (6)$$

These distance measures satisfy the conditions of the metric, and the normalized Euclidean distance has some good geometric properties. But in reality it may not fit so well [22].

Some papers consider distance measure as the dual of similarity concept. Because of the importance of measuring the similarity between IFSs in pattern recognition research, some methods have previously been advanced to calculate the degree of similarity between IFSs [26-28, 32]. In the following parts, these similarity measures are reviewed and discussed. Li and Cheng [34] introduced the similarity measure between the intuitionistic fuzzy sets as below:

Definition 1: Let a mapping s: IFSs(X) × IFSs(X)→[0,1]. s(A,B) is said to be the degree of similarity between $A \in$ IFSs(X) and $B \in$ IFSs(X), if s(A,B) satisfies the following properties:

$$(SP1)\, 0 \le s(A,B) \le 1; \quad (7)$$

$$(SP2)\, If\ A=B, then\ s(A,B)=1; \quad (8)$$

$$(SP3)\, s(A,B)=s(B,A); \quad (9)$$

$$(SP4)\, If\ A \subseteq B \subseteq C, A,B,C \in IFSs(X), \\ then\ s(A,C) \le s(A,B)\ and\ s(A,C) \le s(B,C). \quad (10)$$

But this definition has some limitations. So Mitchell [28] modified it a bit and suggested to replace (SP2) with the 'strong' version (SP2') as follows.

$$(SP2')\quad s(A,B)=1 \Leftrightarrow A=B. \quad (11)$$

With respect to the duality of similarity and distance measure, wang and Xin [22] gave the definition of similarity measure as follows:

Definition 2: Let a mapping d: IFSs(X) × IFSs(X)→[0,1]. d(A,B) is said to be the distance between $A \in$ IFSs(X) and $B \in$ IFSs(X), if d(A,B) satisfies the following properties:

$$(DP1)\, 0 \le d(A,B) \le 1; \quad (12)$$

$$(DP2)\, If\ A=B, then\ d(A,B)=0; \quad (13)$$

$$(DP3)\, d(A,B)=d(B,A); \quad (14)$$

$$(DP4)\, If\ A \subseteq B \subseteq C, A,B,C \in IFSs(X), \\ then\ d(A,C) \ge d(A,B)\ and\ d(A,C) \ge d(B,C). \quad (15)$$

For A and B as IFSs in which $x \in [a,b]$, Li and Cheng [34] proposed a measure for similarity, that is:

$$T_L^c(A,B)=1-\sqrt[p]{\frac{1}{(b-a)}\int_a^b\left[\frac{\left|(\mu_A(x)-\mu_B(x))+(v_A(x)-v_B(x))\right|}{2}\right]^p dx} \quad (16)$$

Which Liu [35] modified it to:

$$T^c(A,B)=1-\sqrt[p]{\frac{1}{2(b-a)}\int_a^b\left[\left|\mu_A(x)-\mu_B(x)\right|^p+\left|v_A(x)-v_B(x)\right|^p+\left|\pi_A(x)-\pi_B(x)\right|^p\right]dx} \tag{17}$$

The degree of similarity between the two IFSs, A and B, can then be calculated as follows [26]:

$$S_d^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{p}\left|m_a(i)-m_B(i)\right|^p} \tag{18}$$

$$\text{Where } m_A(i)=(\mu_A(x_i)+1-v_A(x_i))/2, m_B(i)=(\mu_B(x_i)+1-v_B(x_i))/2, 1\le p<\infty \tag{19}$$

Dengfeng and Chantian [26] extended this measure to the weighted version as well. Liang and Shi [27] proposed the following similarity measures between IFSs. Let $\Phi_{tAB}(i)=\left|\mu_A(x_i)-\mu_B(x_i)\right|/2$ and $\Phi_{fAB}(i)=\left|(1-v_A(x_i))/2-(1-v_B(x_i))/2\right|$. They used $(\Phi_{tAB}(i)+\Phi_{fAB}(i))^p$ to measure the distance between $\left[\mu_A(x_i),1-v_A(x_i)\right]$ and $\left[\mu_B(x_i),1-v_B(x_i)\right]$. Thus, the distance between IFSs A and B was given by:

$$D_e^p(A,B)=\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\Phi_{tAB}(i)+\Phi_{fAB}(i))^p} \tag{20}$$

and $S_e^p(A,B)=1-D_e^p(A,B)$ was defined as a similarity measure between A and B. To get more information on IFSs, Liang and Shi [27] used the middle point $m_A(i)$ to partition the interval $\left[\mu_A(x_i),1-v_A(x_i)\right]$ into two subintervals. One is $\left[\mu_A(x_i),m_A(i)\right]$ and the other is $\left[m_A(i),1-v_A(x_i)\right]$. Then they considered the middle points, say $m_{A1}(i)$ and $m_{A2}(i)$, of these two subintervals. That is,

$$m_{A1}(i)=\frac{\mu_A(x_i)+m_A(i)}{2}, m_{A2}(i)=\frac{m_A(i)+1-v_A(x_i)}{2} \tag{21}$$

Similarly, $m_{B1}(i)$ and $m_{B2}(i)$ are respectively the middle points of subintervals $\left[\mu_B(x_i),m_B(i)\right]$ and $\left[m_B(i),1-v_B(x_i)\right]$, which can be calculated as:

$$m_{B1}(i)=\frac{\mu_B(x_i)+m_B(i)}{2}, m_{B2}(i)=\frac{m_B(i)+1-v_B(x_i)}{2} \tag{22}$$

Thus, they use $(\Phi_{s1}(i)+\Phi_{s2}(i))^p$ to measure the distance between $\left[\mu_A(x_i),1-v_A(x_i)\right]$ and $\left[\mu_B(x_i),1-v_B(x_i)\right]$, where $\Phi_{s1}(i)$ and $\Phi_{s2}(i)$ are given as follows:

$$\Phi_{s1}(i)=\frac{|m_{A1}(i)-m_{B1}(i)|}{2}, \Phi_{s2}(i)=\frac{|m_{A2}(i)-m_{B2}(i)|}{2} \tag{23}$$

The distance between A and B is

$$D_s^p(A,B)=\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}(\Phi_{s1}(i)+\Phi_{s2}(i))^p}. \tag{24}$$

Hence, they used $S_s^p(A,B)=1-D_s^p(A,B)$ to measure the degree of similarity between A and B. In comparison of S_d^p and S_s^p , the formula of S_d^p is simpler than S_s^p , but S_s^p can catch more information in IFSs than S_d^p [33].

Mitchell [28] adopted a statistical approach and interpreted IFSs as ensembles of ordered fuzzy sets to modify Li and Cheng's [34] similarity measure. Let $\rho_\mu(A,B)$ and $\rho_v(A,B)$ denote the similarity measures between the ''low'' membership functions μ_A and μ_A and between the ''high'' membership functions $1-v_A$ and $1-v_B$, respectively, as follows:

$$\rho_\mu(A,B)=S(\mu_A,\mu_B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\mu_A(x_i)-\mu_B(x_i)\right|^p}, \tag{25}$$

$$\rho_v(A,B)=S(1-v_A,1-v_B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|v_A(x_i)-v_B(x_i)\right|^p}. \tag{26}$$

He then defined the modified similarity measure between A and B with

$$S_{\text{mod}}(A,B)=\frac{1}{2}(\rho_\mu(A,B)+\rho_v(A,B)) \tag{27}$$

Hung and Yang [32] suggested some similarity measures between IFSs. First, they used the idea of Hausdorff distance to define the distance between IFSs A and B as follows:

$$d_H(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\{\left|\mu_A(x_i)-\mu_B(x_i)\right|,\left|v_A(x_i)-v_B(x_i)\right|\}. \tag{28}$$

Based on the Hausdorff metric, Grzegorzewski [30] also made generalizations of the Hamming distance, the Euclidean distance and their normalized versions between IFSs. The distance d_H in Hung and Yang [32] is exactly the same as the Hausdorff-type normalized Hamming distance L_h in Grzegorzewski [30]. Furthermore, Hung and Yang [32] used the distance d_H to generate three similarity measures:

$$S_l(A,B)=1-d_H(A,B), \tag{29}$$

$$S_e(A,B)=\frac{\exp(-d_H(A,B))-\exp(-1)}{1-\exp(-1)}, \tag{30}$$

$$S_c(A,B)=\frac{1-d_H(A,B)}{1+d_H(A,B)}. \tag{31}$$

It is known that distance measures and similarity measures are dual concepts. Therefore, Wang and Xin [22] proposed the distance measures between IFSs A and B as follows:

$$d_{wx1}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|v_A(x_i)-v_B(x_i)\right|}{4}+\frac{\max\{\left|\mu_A(x_i)-\mu_B(x_i)\right|,\left|v_A(x_i)-v_B(x_i)\right|\}}{2}\right) \tag{32}$$

And

$$d_{wx2}(A,B)=\frac{1}{\sqrt[p]{n}}\left(\sum_{i=1}^{n}(\Phi_{\mu}(i)+\Phi_{v}(i))^{p}\right)^{1/p}=D_{e}^{p}(A,B), \tag{33}$$

Where $\Phi_{\mu}(i)=|\mu_A(x_i)-\mu_B(x_i)|/2, \Phi_v(i)=|v_A(x_i)-v_B(x_i)|/2.$ (34)

Hung [37] considered an entropy of IFSs induced by the distance LIFS where he made its comparison to Bustince and Burillo [24] and Szmidt and Kacprzyk [36]. To analyze the extent of agreement in a group of experts, Szmidt and Kacprzyk [38] proposed a similarity measure between *A* and *B* based on LIFS (*A,B*). They considered whether *A* is more similar to *B* than to B^c. Their proposed measure, S_{sk} (*A,B*), can answer the question if *A* is more similar or more dissimilar to *B*, where

$$S_{sk}(A,B)=L_{IFS}(A,B^c)-L_{IFS}(A,B). \tag{35}$$

Obviously, $-1\le S_{sk}(A,B)\le 1$ and $S_{sk}(A,B)=S_{sk}(B,A).$ (36)

In comparison of d_H of Hung and Yang [32], L_h of Grzegorzewski [30] and LIFS of Szmidt and Kacprzyk [36], they all have the property that d_H(A,B)= L_h(A,B)= L_{IFS}(A,B)=0 if A=B. However, Szmidt and Kacprzyk [36] and Grzegorzewski [30] only discussed distances between IFSs. But, similarity measures are very useful in areas, such as pattern recognition, machine learning and decision making. Szmidt and Kacprzyk [38] proposed a similarity measure S_{sk} based on L_{IFS} and Hung and Yang [32] created similarity measures S_l, S_e and S_c based on the distance d_H.

Later, Hung and Yang [33] proposed a distance and similarity measure based on L_P distance between two interval. They defined the similarity measures between IFSs A and B as follows:

$$S_l^p(A,B)=\frac{2^{1/p}-L_p(A,B)}{2^{1/p}}, \tag{37}$$

$$S_e^p(A,B)=\frac{\exp(-L_p(A,B))-\exp(-2^{1/p})}{1-\exp(-2^{1/p})}, \tag{38}$$

$$S_c^p(A,B)=\frac{2^{1/p}-L_P(A,B)}{2^{1/p}(1+L_P(A,B))}. \tag{39}$$

where

$$I_A(x_i)=\left[\mu_A(x_i),1-v_A(x_i)\right],\quad i=1,2,...,n, \tag{40}$$

$$d_p(I_A(x_i),I_B(x_i))=(|\mu_A(x_i)-\mu_B(x_i)|^p+|v_A(x_i)-v_B(x_i)|^p,p\ge 1, \tag{41}$$

$$L_p(A,B)=\frac{1}{n}\sum_{i=1}^{n}d_p(I_A(x_i),I_B(x_i)). \tag{42}$$

3 Proposed measures

As discussed in previous section, there exist some similarity or dissimilarity measures between IFSs in literature which can be used for pattern recognition; but existent measures have some shortages. In each of these measures, distance or similarity is calculated from a fix sets (patterns) and there is not a measure which could evaluate distance/similarity from/to a set with membership/non-membership function instead of fixed values as membership/non-membership. This paper proposes a more generalized definition of distance measures between IFSs, in which the degrees of membership and non-membership are functions of an indicator, not fixed values. This kind of membership is considered [26] but not in the manner presented in this paper. Assume that, there are three primal patterns: child, young and old; and the degree of membership and non-membership of a person is a function from his/her age. Thus, this person belongs to the group which the distance between membership/non-membership degrees obtained from the mathematical function and the membership/non-membership degrees that experts assigned to it, is the smallest.

Hence, some changes to Hamming and Euclidean distance proposed by Szmidt and Kacprzyk [29] are addressed: According to this theory, we consider pattern recognition process as follows: Consider P={1, 2, …, p} as a set of different patterns (groups) which cases would be assigned to. Now assume $\mu_A^\alpha(x_i)$ and $\nu_A^\alpha(x_i)$ are respectively representing membership and non-membership functions of case x_i to pattern α; in addition $\mu_B^\alpha(x_i)$ and $\nu_B^\alpha(x_i)$ indicate membership and non-membership degrees to pattern α based on experts' opinion. x_i would be assigned to pattern α, if the distance between A and B within pattern α is less than similar distance for any other pattern in set P.

In order to calculate this distance, the proposed measures [29] have been modified to the presented formula in equations 43-46. Furthermore, the proposed distance formulations have another unique characteristic which has not been considered in previously developed models in literature. In pattern recognition literature of IFSs, the base pattern is fixed and the up-dated earned knowledge from new cases of each pattern is ignored. When case x_i is assigned to pattern α, the knowledge on this pattern has been increased. So, it is wise to employ the obtained knowledge and improve the recognition progress. By neglecting this new information, case x_{i+1} will be compared with the first pre-defined pattern but we suggest comparing it with center of gravity (median point) for all the previously assigned cases to each group and the base pattern. Consequently, the increased knowledge is captured in the proposed measures (equations 43-46) by introducing $\overline{\mu}_A(x_i)$, $\overline{\nu}_A(x_i)$ and $\overline{\pi}_A(x_i)$. In order to satisfy all these issues, Hamming, Euclidean, normalized Hamming and Euclidean distance are formulated as follows:

Let $A=\{(x,\mu_A(x),\nu_A(x))|x\in X\}$, $B=\{(x,\mu_B(x),\nu_B(x))|x\in X\}$ be two IF sets in X = {x_1, x_2, . . . , x_n}. Now we can propose:

Hamming distance:

$$d_h(A,B)=\frac{1}{2}\sum_{i=1}^{n}(|\overline{\mu}_A(x_i)-\mu_B(x_i)|+|\overline{\nu}_A(x_i)-\nu_B(x_i)|+|\overline{\pi}_A(x_i)-\pi_B(x_i)|) \quad (43)$$

Euclidean distance:

$$d_e(A,B)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}(\overline{\mu}_A(x_i)-\mu_B(x_i))^2+(\overline{\nu}_A(x_i)-\nu_B(x_i))^2+(\overline{\pi}_A(x_i)-\pi_B(x_i))^2} \quad (44)$$

Normalized Hamming distance:

$$d_h^n(A,B)=\frac{1}{2n}\sum_{i=1}^{n}(\left|\overline{\mu}_A(x_i)-\mu_B(x_i)\right|+\left|\overline{v}_A(x_i)-v_B(x_i)\right|+\left|\overline{\pi}_A(x_i)-\pi_B(x_i)\right|) \tag{45}$$

Normalized Euclidean distance:

$$d_e^n(A,B)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}(\overline{\mu}_A(x_i)-\mu_B(x_i))^2+(\overline{v}_A(x_i)-v_B(x_i))^2+(\overline{\pi}_A(x_i)-\pi_B(x_i))^2} \tag{46}$$

where:

$$\overline{\mu}_A(x_i)=\frac{1}{m}\sum_{j=1}^{m}\mu_{Aj}(x_i) \tag{47}$$

$$\overline{v}_A(x_i)=\frac{1}{m}\sum_{j=1}^{m}v_{Aj}(x_i) \tag{48}$$

$$\overline{\pi}_A(x_i)=\frac{1}{m}\sum_{j=1}^{m}\pi_{Aj}(x_i) \tag{49}$$

and m indicates the number of cases, classified to the base pattern previously.

Proposition. The proposed normalized Hamming distance (equ. 45) and normalized Euclidean distance (equ. 46) satisfy the distance measures conditions in equ. 12-15 [22].

Proof. Obviously, $d_h^n(A,B)$ and $d_e^n(A,B)$ satisfy (DP1-DP3) of equ. 12-15. We only need to prove $d_h^n(A,B)$ and $d_e^n(A,B)$ satisfy (DP4).

For any IFSs $C=\{(x,\mu_C(x),v_C(x))\,|\,x\in X\}$ if $\overline{A}\subseteq B\subseteq C$, then it is easy to see that

$$\left|\overline{\mu}_A(x_i)-\mu_C(x_i)\right|\geq\left|\overline{\mu}_A(x_i)-\mu_B(x_i)\right| ,$$

$$\left|\overline{v}_A(x_i)-v_C(x_i)\right|\geq\left|\overline{v}_A(x_i)-v_B(x_i)\right| \text{ and}$$

$$\left|\overline{\pi}_A(x_i)-\pi_C(x_i)\right|\geq\left|\overline{\pi}_A(x_i)-\pi_B(x_i)\right| .$$

So we have

$$\frac{1}{2n}\sum_{i=1}^{n}(\left|\overline{\mu}_A(x_i)-\mu_C(x_i)\right|+\left|\overline{v}_A(x_i)-v_C(x_i)\right|+\left|\overline{\pi}_A(x_i)-\pi_C(x_i)\right|$$
$$\geq\frac{1}{2n}\sum_{i=1}^{n}(\left|\overline{\mu}_A(x_i)-\mu_B(x_i)\right|+\left|\overline{v}_A(x_i)-v_B(x_i)\right|+\left|\overline{\pi}_A(x_i)-\pi_B(x_i)\right|))$$

and then we can get the inequality: $d_h^n(\bar{A},C) \geq d_h^n(\bar{A},B)$. By the same reason we can get $d_h^n(\bar{A},C) \geq d_h^n(B,C)$. Therefore $d_h^n(A,B)$ satisfies (DP4) of equ. 12-15. By the same way we can prove satisfaction of properties in equ. 12-15 for $d_e^n(A,B)$.

In addition, it is provable that $S_h^n(A,B) = 1 - d_h^n(A,B)$ and $S_e^n(A,B) = 1 - d_e^n(A,B)$ which satisfy properties in equ. 7-11. So, the proposed equations can be called similarity measures.

The given proof supports the theoretical validity of distance measures. It was proved that the proposed formulations hold required characteristics of distance measures so they can be employed in pattern recognition process. But in addition to the theoretical backbone of the addressed models, this paper checks their reliability through performing appropriate experiments in a medical diagnosis process. In next sections the proposed measures will be applied in a practical problem of bacillus colony recognition, and the results will be compared with regular fuzzy.

4 Application of proposed measures in practice

In medical diagnosis of laboratories there are almost always some primal diagnosis based on the shape of case. The final judgment depends on additional investigations and biochemical tests like amount of growth in a given time or reaction with special elements or generating gas. Nowadays, some software can detect some of primal factor for instance counting number of particles, but experts indicate that the software cannot support some cases and often present unacceptable results. Most of the analysis processes in medical labs are inherently illustrated by linguistic variables like oval shape, spiral shape, rather big and so on. In this process, the diagnosis is naturally uncertain, so fuzzy approach can be addressed to investigate it; on the other hand, this uncertainty is not just in judgment but also in the truth of this identification. Hence, there is a need for a kind of fuzzy which could support the latter uncertainty. IFS is appropriate model for its formulation which was illustrated in section 3. In medical laboratories the non-membership degree of IFSs is a significant concept which can be used in formulation of pattern recognition progress of some circumstances.

This paper attempts to formulate a part of diagnosis in laboratories which is colonies recognition. We can address this issue as an example of pattern recognition problem. In order to run the experiments and evaluate the proposed pattern recognition process and distance measures, three experts in microbiology department of a medical laboratory were asked to involve in this research process. They did not have any background knowledge on fuzzy logic and more specifically on intuitionistic fuzzy. So, the general idea and logic of IFSs were elaborated during first meeting. They became familiar with the concept of membership and non-membership to be able to discuss on given numbers in table 1 as well as equations 50-57. The four investigated bacilli in this paper are also based on their suggestions. First of all they were asked to pinpoint the visual criteria for assigning a case to each pattern. They specified five different criteria: (1) domical shape in macro level (on eye vision) of the colony for each pattern, (2) being coccus (single microscopic shape), (3) being diplococcus (double microscopic shape), (4) existence of flagellum, and (5) size of colony. All the four examined bacilli are gram-negative and cultivated in similar culture medium. A typical case of each pattern holds specific value of membership and non-membership for the first four criteria. Table 1 exhibits these values which have been assigned by experts.

Table 1. Features of each pattern

Patterns	Domical shape		Coccus		Diplococcus		Flagellum	
	μ	v	μ	v	μ	v	μ	v
Bacil Coli	0.9	0.05	0.9	0	0	1	0.9	0.06
Shigella	0.9	0.08	0.9	0.05	0.05	0.92	0.08	0.9
Salmonella	0.8	0.1	0.8	0.1	0.1	0.85	0.9	0.01
Klib Siella	0.8	0.15	0.7	0.15	0.2	0.75	0.1	0.85

The last criterion is size of colony; in which the membership and non-membership degrees of a case to each pattern should be calculated through a mathematical function based on the measured size. In order to find these functions, experts were asked to demonstrate how size of colony for each pattern would vary. The range of their size can also be found in microbiology text books but such details are not related to our work. So, experts were also asked some follow up questions for each pattern, e.g. what is the minimum and maximum acceptable colony size? What is the most common colony size? If you see a colony of bacilli with size x, how much do you believe that it belongs to each pattern and how much you believe that it does not belong to it? Based on the answers to the two first questions, triangular and bell shape of membership and non-membership functions were developed and checked for correctness based on the third question. Triangular and bell shape functions returned membership and non-membership values of a colony with size x, which we compared with what experts assign to them. A number of these comparisons revealed significant mismatch. Consequently, we tried to develop appropriate membership and non-membership functions based on experts' answers to aforementioned questions. These functions were fitted to the given numbers by experts. Equations 50-57 represent membership and non-membership functions for Bacil-Coli, Shigella, Salmonella and Klib-Siella, respectively. Suggested functions are tent-shape and figures 1-4 shows them graphically.

In order to evaluate the effectiveness of proposed distance measures and pattern recognition process, sixty cases were studied. Experts were asked to give size of colony for each case as well as membership and non-membership degree of them to each pattern based on illustrated criteria. Based on the given values in table 1 and equations 50-57, $\mu_A(x_i)$ and $\nu_A(x_i)$ were calculated as the base pattern; and given numbers by experts represented $\mu_B(x_i)$ and $\nu_B(x_i)$ in equations 43-46. According to the proposed distance measure, we could assign each case to one of the patterns, and experts could also assign them to the right pattern based on their observation without bringing our calculations into account. So, for a given patient the cultivated bacteria in medium had specific visual features which led experts to identification of the bacteria. The proposed formulation also took similar steps but by translating visual observations to numbers. What we call here as right pattern is the correct identification of bacteria which was determined by experts.

$$\begin{matrix}\mu(x)\\ \text{for Bacil}-\text{Coli}\end{matrix} = \begin{cases} 6x-11.4 & 1.9 \le x < 2 \\ 0.6x-0.6 & 2 \le x < 2.5 \\ -0.6x+2.4 & 2.5 \le x < 3 \\ -6x+18.6 & 3 \le x < 3.1 \\ 0 & e.w. \end{cases} \tag{50}$$

$$\begin{matrix}\nu(x)\\ \text{for Bacil}-\text{Coli}\end{matrix} = \begin{cases} -3.5x+7.3 & 1.8 \le x < 2 \\ -0.6x+1.5 & 2 \le x < 2.5 \\ 0.6x-1.5 & 2.5 \le x < 3 \\ 3.5x-10.2 & 3 \le x < 3.2 \\ 0 & e.w. \end{cases} \tag{51}$$

$$\begin{matrix}\mu(x)\\ \text{for Shigella}\end{matrix} = \begin{cases} 6x-5.4 & 0.9 \le x < 1 \\ 0.6x & 1 \le x < 1.5 \\ -0.6x+1.8 & 1.5 \le x < 2 \\ -6x+12.6 & 2 \le x < 2.1 \\ 0 & e.w. \end{cases} \tag{52}$$

$$\begin{matrix}\nu(x)\\ \text{for Shigella}\end{matrix} = \begin{cases} -3.5x+3.8 & 0.8 \le x < 1 \\ -0.6x+0.9 & 1 \le x < 1.5 \\ 0.6x-0.9 & 1.5 \le x < 2 \\ 3.5x-6.7 & 2 \le x < 2.2 \\ 0 & e.w. \end{cases} \tag{53}$$

$$\underset{for\ Salmonella}{\mu(x)} = \begin{cases} 6x-11.4 & 1.9 \le x < 2 \\ 0.6x-0.6 & 2 \le x < 2.5 \\ -0.6x+2.4 & 2.5 \le x < 3 \\ -6x+18.6 & 3 \le x < 3.1 \\ 0 & e.w. \end{cases} \tag{54}$$

$$\underset{for\ Salmonella}{\nu(x)} = \begin{cases} -3.5x+7.3 & 1.8 \le x < 2 \\ -0.6x+1.5 & 2 \le x < 2.5 \\ 0.6x-1.5 & 2.5 \le x < 3 \\ 3.5x-10.2 & 3 \le x < 3.2 \\ 0 & e.w. \end{cases} \tag{55}$$

$$\underset{for\ Klib-Siella}{\mu(x)} = \begin{cases} 6x-11.4 & 1.9 \le x < 2 \\ 0.3x & 2 \le x < 3 \\ -0.3x+1.8 & 3 \le x < 4 \\ -6x+24.6 & 4 \le x < 4.1 \\ 0 & e.w. \end{cases} \tag{56}$$

$$\underset{for\ Klib-Siella}{\nu(x)} = \begin{cases} -3.5x + 7.3 & 1.8 \le x < 2 \\ -0.3x+0.9 & 2 \le x < 3 \\ 0.3x-0.9 & 3 \le x < 4 \\ 3.5x-13.7 & 4 \le x < 4.2 \\ 0 & e.w. \end{cases} \tag{57}$$

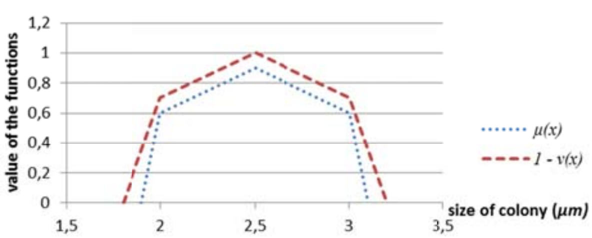

Figure 1. membership and non-membership functions for size of colony (Bacil-Coli)

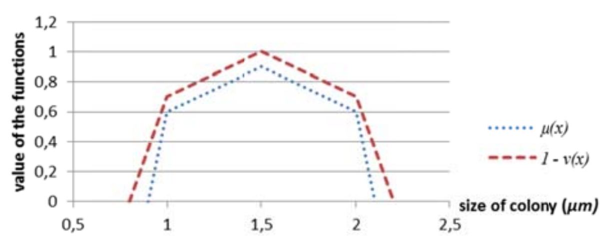

Figure 2. membership and non-membership functions for size of colony (Shigella)

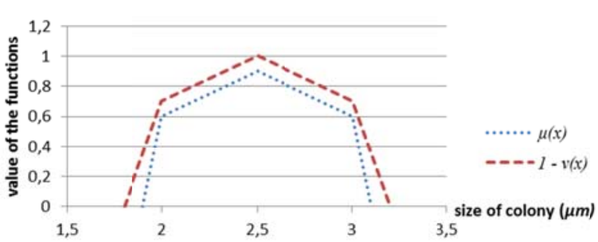

Figure 3. membership and non-membership functions for size of colony (Salmonella)

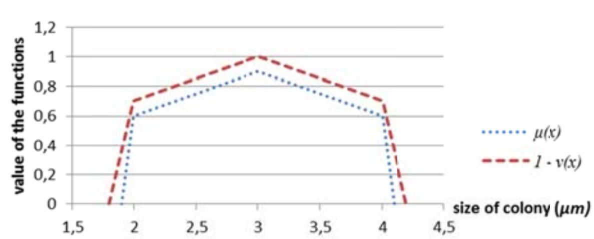

Figure 4. membership and non-membership functions for size of colony (Klib-Siella)

As illustrated in section 3, the proposed procedure would assign each case to the pattern with minimum distance between the base pattern and expert idea. By this logic, suggested formulation could assign 94% of cases to the right pattern. Table 2, summarizes the calculated distances for wrongly assigned cases. For these cases, the calculated distance from the right pattern is more than the distance from the wrong pattern. As table 2 exhibits, this error is in the range of [0.01, 0.11] for Hamming and Euclidean distance and in the range of [0.002, 0.03] for normalized distance measures. This means that the distance from the right pattern is more than the distance from the wrong pattern and this difference is maximum 0.03 for normalized distances. According to the level of accuracy for significant percentage of investigated cases, the occurred error is acceptable. It is worth to mention that formulated problem is the first step of laboratory diagnosis process, and further tests should be done to finalize the diagnosis. So, such errors can happen in real environment as well which would be verified by complementary tests.

Table 2. Comparison of calculated distances for wrongly assigned cases

Hamming distance						**Normalized Hamming distance**					
no	Bacil-Coli	Shigella	Salmonella	Klib-Siella	error	no.	Bacil-Coli	Shigella	Salmonella	Klib-Siella	error
24	1,318	0,630	1,723	0,660	0,030	24	0,264	0,126	0,345	0,132	0,006
25	0,720	1,195	0,830	1,462	0,110	25	0,144	0,239	0,166	0,292	0,022
47	0,530	1,347	0,590	1,463	0,060	47	0,106	0,269	0,118	0,293	0,012
51	0,550	1,403	0,540	1,228	0,010	51	0,110	0,281	0,108	0,246	0,002
Euclidean distance						**Normalized Euclidean distance**					
no	Bacil-Coli	Shigella	Salmonella	Klib-Siella	error	no.	Bacil-Coli	Shigella	Salmonell a	Klib-Siella	error
25	0,360	0,732	0,390	0,769	0,030	25	0,161	0,328	0,174	0,344	0,013

5 Comparison with regular fuzzy

Based on the discussion on previous section, it is concluded that the proposed measures result in acceptable answers, but we need to show if IFSs approach is more precise than regular fuzzy and previous proposed measures in literature. Regular fuzzy ignores the hesitancy of experts and assumes $\nu = 1 - \mu$. So, by substituting ν with $1 - \mu$, all the formulations of IFSs would be equivalent to regular fuzzy. In order to perform this comparison, the process of pattern recognition based on proposed distance measures has been done again for all the investigated cases in section 4, regarding the change in distance measure to regular fuzzy. In this examination four principles were designed to make the comparison more precise and valid. Assume:

DF_r: the calculated distance of a case from the right pattern in regular fuzzy,

DIF_r: the calculated distance of a case from the right pattern in intuitionistic fuzzy,

DF_w: the calculated distance of a case from the wrong pattern in regular fuzzy,

and

DIF_w: the calculated distance of a case from the wrong pattern in intuitionistic fuzzy.

By considering these definitions, occurrence of following conditions indicates the advantages of IFSs in comparison to regular fuzzy:

$$DF_w < DF_r \text{ and } DIF_r < DIF_w \quad (58)$$

$$DF_w < DIF_w \ and \ DIF_r < DF_r \quad (59)$$

$$if \ DIF_w < DF_w and \ DIF_r < DF_r \quad (60)$$

$$then \ DF_w - DIF_w < DF_r - DIF_r \Rightarrow \frac{DF_w - DIF_w}{DF_r - DIF_r} < 1 \Rightarrow compare \ scale = \frac{DF_w - DIF_w}{DF_r - DIF_r} - 1 < 0$$

$$if \ DF_w < DIF_w and \ DF_r < DIF_r \quad (61)$$

$$then \ DF_r - DIF_r < DF_w - DIF_w \Rightarrow \frac{DF_r - DIF_r}{DF_w - DIF_w} < 1 \Rightarrow compare \ scale = \frac{DF_w - DIF_w}{DF_r - DIF_r} - 1 > 0$$

The first item reflects the situation where the studied case is assigned to the wrong pattern in regular fuzzy but the formulation on IFSs return the correct assigned pattern. The second item reflects the condition when the calculated distance by regular fuzzy approach is higher than IFSs from the right pattern and lower than IFSs from the wrong pattern. The third item is valid when the distance based on regular fuzzy approach is higher than IFSs from the right pattern and also from the wrong pattern, but this difference is more in right pattern and less in wrong patterns. Finally the forth item covers the situations where the obtained distance of regular fuzzy approach is lower than IFSs from the right pattern and also from the wrong pattern, but this reduction is less from right pattern and more from wrong patterns.

If the results indicate the first principle, it means regular fuzzy approach does not give correct answer while intuitionistic fuzzy does. And if the next three principles occur, it can be concluded that intuitionistic fuzzy approach results in more accurate answers. Tables in Appendix exhibit the calculated distances based on both intuitionistic and regular fuzzy approach as well as the comparison scale values. According to the given numbers in tables A.1, A.2, A.3 and A.4 in appendix, IFS represents more accurate results than regular fuzzy in more than 90% cases; for example the fourth row of Table A.3 in Appendix shows computational result for a case belongs to Klib-Siella pattern based on calculated normalized Hamming distance, the compared value shows that the distance calculated in IFS is more accurate than in fuzzy approach, in this case the obtained distance from regular fuzzy approach is less than IFSs from the right pattern and wrong pattern, this reduction is less in right pattern and more in wrong patterns; and this comparison is measured by compare scale.

Through the same way we can find the first row of Table A.4 in Appendix, this case belongs to Bacil-Coli pattern based on calculated normalized Euclidean distance, the value of compare scale shows that the distance calculated in IFS is more accurate than in fuzzy approach, in this case the obtained distance from regular fuzzy approach is higher than IFSs from the right pattern and also from the wrong pattern, this rise is more in right pattern and less in wrong patterns; and this comparison is measured by compare scale. Besides, regular fuzzy return wrong answer in greyed cells in which IFS present right outcome. Of course, as mentioned before, there are some cases with wrong answer in IFS, but these cases have incorrect answer with regular fuzzy too. According to the illustrated comparison, advantages of intuitionistic fuzzy over regular fuzzy is supported, which also shows the proposed measures in IFSs are more appropriate than regular fuzzy application in medical diagnosis; and can be mentioned as valid distance measures for IFSs.

6 Practical implications

Theoretical analysis of fuzzy systems and pattern recognition are interesting topics for researchers. Considerable number of papers have been published in these fields and researchers tried to expand boarders of knowledge and theories. Apart from theoretical aspects of such research, their implication to real environment is also required to be clear. This paper provided an example of employing IF pattern recognition in practice by performing different experiments in the process of medical laboratory diagnosis. Pattern recognition is among the hot topics in artificial intelligence and one of its important applications is in medical diagnosis. The main aim of such research is to develop required models to facilitate programing machines which can be done part of diagnosis or can be employed to help experts in confirming their decision.

The proposed distance measures and pattern recognition process are not just limited to medical diagnosis. Face detection, hand writing identification and finger print analysis can also be considered as other applications. What we developed and illustrated here was bounded to a very limited part of the whole diagnosis process. This starting block can be used to build up further steps. No one can claim that by implementing such techniques, experts can be replaced by machines to make decision instead of them in close future. But all the research in this field, including this paper, tries to pave the road to facilitate assistance of machines in human decision making process.

7 Conclusion

This paper studied pattern recognition process when cases and their characteristics can be interpreted based on intuitionistic fuzzy sets theory. More specifically, the main focus of this paper was on employed distance measurement techniques in pattern recognition process. Based on the required improvement in existing distance measurement models in literature, four distance measurement model were addressed. For the purpose of checking validity and reliability of developed formulations, on step of medical diagnosis in laboratories was studied. The performed experiments were done in microbiology department of a medical laboratory and three experts were asked to participate in this study. Sixty cases were examined and 94% of them correctly assigned to the right pattern, while the rest 6% had very small error in resulted outcomes. This ratio of correctness supported the reliability of the proposed measures. Furthermore, pattern recognition experiments were repeated by formulations for regular fuzzy which revealed the preference and preciseness of intuitionistic fuzzy approach over regular fuzzy.

In addition to employing the proposed distance models in real environment, the addressed distance measures covered some of the shortages in existing models of literature. First of all, previous measurement formulations calculated distance or similarity from a fixed set (pattern). But the addressed models of this paper are capable of evaluating distance/similarity from/to a set with membership/non-membership function instead of fixed values as membership/non-membership. Secondly, the developed distance measurement models tried to capture knowledge of assigning a case to a pattern. This knowledge has been neglected in existing models of literature. So, this paper introduced median or gravity point of previously assigned case to one specific pattern and used it in the formulation of distance measurement.

For future research, two streams of theoretical and practical expansion are suggested. The proposed measurement formulation assumed equal weight for all of the characteristics, so the expansion of proposed models to weighted distance measurement can be one of the future directions for this research. Furthermore, it is suggested to expand the process of diagnosis to further steps and to more advanced applications. One application can be in the process of cancer diagnosis where different indicators can be formulated in membership and non-membership functions.

Acknowledgment

Authors would like to thank two anonymous reviewers for their valuable inputs to improve the quality of this paper. This research could not be done without the support of Resalat Medical Laboratory and its collaborative employees; authors would like to thank you all.

References

[1] R. L. Gregory, Ed., The Oxford Companion to the Mind. Oxford, UK: Oxford University Press, 2004.
[2] H. B. Mitchell, "Pattern recognition using type-II fuzzy sets," Information Sciences. 2005; 170: 409-418. http://dx.doi.org/10.1016/j.ins.2004.02.027
[3] G. J. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic: Prentice-Hall, Upper Saddle River, 1995.

[4] C.-M. Hwang, M.-S. Yang, W.-L. Hung, and M.-G. Lee, "A similarity measure of intuitionistic fuzzy sets based on the Sugeno integral with its application to pattern recognition," Information Sciences. 2012; 189: 93-109. http://dx.doi.org/10.1016/j.ins.2011.11.029
[5] S. Mitra and S. K. Pal, "Fuzzy sets in pattern recognition and machine intelligence," Fuzzy Sets and Systems. 2005; 156: 381-386. http://dx.doi.org/10.1016/j.fss.2005.05.035
[6] T. Twellmann, A. Meyer-Baese, O. Lange, S. Foo, and T. W. Nattkemper, "Model-free visualization of suspicious lesions in breast MRI based on supervised and unsupervised learning," Engineering Applications of Artificial Intelligence. 2008; 21: 129-140. PMid:19255616 http://dx.doi.org/10.1016/j.engappai.2007.04.005
[7] J. G. Wolff, "Medical diagnosis as pattern recognition in a framework of information compression by multiple alignment, unification and search," Decision Support Systems. 2006; 42: 608 - 625. http://dx.doi.org/10.1016/j.dss.2005.02.005
[8] M. R. Ogiela and R. Tadeusiewicz, "Artificial intelligence structural imaging techniques in visual pattern analysis and medical data understanding," Pattern Recognition. 2003; 36: 2441-2452. http://dx.doi.org/10.1016/S0031-3203(03)00089-X
[9] A. Meyer-Bäse, "Neuro-fuzzy classification," Pattern Recognition in Medical Imaging. 2004: 282-317.
[10] L. I. Kuncheva, R. Z. Zlatev, S. N. Neshkova, and H. Gamper, Eds., Fuzzy Logic in Artificial Intelligence (Lecture Notes in Computer Science. Berlin / Heidelberg: Springer; 2006.
[11] Z. Zhang, J. Yang, Y. Ye, Y. Hu, and Q. Zhang, "A Type of Score Function on Intuitionistic Fuzzy Sets with Double Parameters and Its application to Pattern Recognition and Medical Diagnosis," Procedia Engineering. 2012; 29: 4336-4342. http://dx.doi.org/10.1016/j.proeng.2012.01.667
[12] I. K. Vlachos and G. D. Sergiadis, "Intuitionistic fuzzy information: Applications to pattern recognition," Pattern Recognition Letters. 2007; 28: 197-206. http://dx.doi.org/10.1016/j.patrec.2006.07.004
[13] N. Karnik and J. M. Mendel. "Operations on type-2 fuzzy sets," Fuzzy Sets and Systems. 2001; 122: 327-348, 2001. http://dx.doi.org/10.1016/S0165-0114(00)00079-8
[14] J. M. Mendel and R. I. John, "Type-2 Fuzzy Sets Made Simple," IEEE Transaction on Fuzzy Systems. 2002; 10: 117-127, 2002. http://dx.doi.org/10.1109/91.995115
[15] G. Deschrijver and P. Kral, "On the cardinalities of interval-valued fuzzy sets," Fuzzy Sets and Systems. 2007; 158: 1728-1750, 2007. http://dx.doi.org/10.1016/j.fss.2007.01.005
[16] W. Zeng and H. Li, "Relationship between similarity measure and entropy of interval valued fuzzy sets," Fuzzy Sets and Systems. 2006; 157: 1477-1484, 2006. http://dx.doi.org/10.1016/j.fss.2005.11.020
[17] K. T. Atanassov, "Two theorems for intuitionistic fuzzy sets," Fuzzy Sets and Systems. 2000; 110: 267-269. http://dx.doi.org/10.1016/S0165-0114(99)00112-8
[18] H. Bustince, "Construction of intuitionistic fuzzy relations with predetermined properties," Fuzzy Sets and Systems. 2000; 109: 379-403. http://dx.doi.org/10.1016/S0165-0114(97)00381-3
[19] W. Zeng and H. Li, "Correlation coefficient of intuitionistic fuzzy sets," Journal of Industrial Engineering International. 2007; 3: 33-40.
[20] J. Li, G. Deng, H. Li, and W. Zeng, "The relationship between similarity measure and entropy of intuitionistic fuzzy sets," Information Sciences. 2012; 188: 314-321. http://dx.doi.org/10.1016/j.ins.2011.11.021
[21] D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk, and H. Prade, "Terminological difficulties in fuzzy set theory-The case of Intuitionistic Fuzzy Sets," Fuzzy Sets and Systems. 2005; 156: 485-491, 2005. http://dx.doi.org/10.1016/j.fss.2005.06.001
[22] W. Wang and X. Xin, "Distance measure between intuitionistic fuzzy sets," Pattern Recognition Letters. 2005; 26: 2063-2069.
[23] C. Cornelis, G. Deschrijver, M. D. Cock, and E. Kerre, "Intuitionistic Fuzzy Relational Calculus: an Overview," Intelligent Systems, 2002 Proceedings. First International IEEE Symposium. 2002; 1: 340-345.
[24] H. Bustince and P. Burillo, "Vague sets are intuitionistic fuzzy sets," Fuzzy Sets Systems. 1996; 79: 403-405. http://dx.doi.org/10.1016/0165-0114(95)00154-9
[25] Y. Li, D. L. Olson, and Z. Qin, "Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis," Pattern Recognition Letters. 2007; 28: 278-285, 2007. http://dx.doi.org/10.1016/j.patrec.2006.07.009
[26] L. Dengfeng and C. Chantian, "New similarity measures of intuitionistic fuzzy sets and applications to pattern recognitions," Pattern Recognition Letters. 2002; 23: 221-225, 2002. http://dx.doi.org/10.1016/S0167-8655(01)00110-6
[27] Z. Liang and P. Shi, "Similarity measures on intuitionistic fuzzy sets," Pattern Recognition Letters. 2003; 24: 2687-2693. http://dx.doi.org/10.1016/S0167-8655(03)00111-9
[28] H. B. Mitchell, "On the Dengfeng-Chuitian similarity measure and its application to pattern recognition," Pattern Recognition Letters. 2003; 24: 3101-3104, 2003. http://dx.doi.org/10.1016/S0167-8655(03)00169-7
[29] E. Szmidt and J. Kacprzyk, "Distances between intuitionistic fuzzy sets," Fuzzy Sets and Systems. 2000; 114: 505-518. http://dx.doi.org/10.1016/S0165-0114(98)00244-9

[30] P. l. Grzegorzewski, "Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdor& metric," Fuzzy Sets and Systems. 2004; 148: 319-328, 2004. http://dx.doi.org/10.1016/j.fss.2003.08.005

[31] D.-F. Li, "Some measures of dissimilarity in intuitionistic fuzzy structures," Journal of Computer and System Sciences. 2004; 68: 115-122.

[32] W.-L. Hung and M.-S. Yang, "Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance," Pattern Recognition Letters. 2004; 25: 1603-1611.

[33] W.-L. Hung and M.-S. Yang, "Similarity measures of intuitionistic fuzzy sets based on Lp metric," International Journal of Approximate Reasoning. 2007; 46: 120-136. http://dx.doi.org/10.1016/j.ijar.2006.10.002

[34] D. F. Li and C. T. Cheng, "New similarity measures of intuitionistic fuzzy sets and application to pattern recognition," Pattern Recognition Letters. 2002; 23: 221-225. http://dx.doi.org/10.1016/S0167-8655(01)00110-6

[35] H.-W. Liu, "New Similarity Measures Between Intuitionistic Fuzzy Sets and Between Elements," Mathematical and Computer Modelling. 2005; 42: 61-70. http://dx.doi.org/10.1016/j.mcm.2005.04.002

[36] E. Szmidt and J. Kacprzyk, "Entropy of intuitionistic fuzzy sets," Fuzzy Sets and Systems. 2001; 118: 467-477. http://dx.doi.org/10.1016/S0165-0114(98)00402-3

[37] W. L. Hung, "A note on entropy of intuitionistic fuzzy sets," International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 11, pp. 627-633, 2003. http://dx.doi.org/10.1142/S0218488503002375

[38] E. Szmidt and J. Kacprzyk, "A new concept of a similarity measure for intuitionistic fuzzy sets and its use in group decision making," 2005; 3558 LNAI, ed: 272-282.